DEGREE LEVEL

REAL ANALYSIS

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Degree Level Real Analysis

(For Universities of Bihar)

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Dedicated

to

the loving memory of

Late Mohit Mohan Bose

an exemplary personality of honesty, foresight and efficiency whose inspiration made us what we are to-day.

Das Gupta : Prasad

PREFACE

The book is designed to serve as a text-book for Degree Level students and is written according to the new syllabus of different Universities of Bihar for "First two years of plus three course". Many of the students, after completing two years, will go in for Honours Course in the third year. As the same topics will not be taught again in the third year, it is desirable that the book should not be too easy or too advanced to the mixed group of undergraduate students.

Some of the salient features of the book are the following—

(i) All essential matters have been discussed with as much thoroughness and precision as possible. We have also tried to be rigorous in treatment without being complicated.

(ii) Different definitions and concepts have been introduced with suitable examples.

(iii) Throughout the book emphasis has been laid on different types of examples. A large number of problems have been worked out in graded order. Hints to the unsolved problems have been given at the end. Problems set in different University and competitive examinations have been incorporated. Constructive criticisms and helpful suggestions for the improvement of the book will be gratefully appreciated.

August, 1984

Authors

Second Edition

The first edition of the book has been well received by the readers. The present edition incorporates the changes suggested by teachers as well as the students.

We hope, this new edition should be more useful to the readers.

March, 1988

Authors

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List of Symbols and Results

1. Symbols and their meaning—

Symbol	*Meaning*
N	the set of natural numbers
R	the set of real number
Q	the set of rational numbers
Z	the set of integers
P	the set of irrational numbers
$x \in R$	x belongs to R (or x is a real number)
$\Rightarrow$	implies
$\Leftrightarrow$	if and only if (iff)
$\forall$	for all
$\exists$	there exists
$\mid x \mid$	the absolute value of x i.e. $\mid x \mid = x$ when $x \geqslant 0$ $-x$ when $x \leqslant 0$

Results to remember for working out problems—

2. (i) $|a-b| < \epsilon \Leftrightarrow b-\epsilon < a < b+\epsilon$.
   (ii) $k \in R^+$ and $|a-b| < k.\epsilon \ \forall \ \epsilon > 0 \Rightarrow a = b$.
   (iii) $k \leqslant x \leqslant K \ \forall \ x \in F \Leftrightarrow F$ is bounded and k, K are respectively a lower and an upper bound.
   (iv) $x \in (l-\epsilon,\ l+\epsilon) \Leftrightarrow l-\epsilon < x < l+\epsilon$.

3, (i) u_n is bounded $\Leftrightarrow u_n \leqslant K$ for all $n \in N$, $K \in R$.
   (ii) M is the upper bound (or least upper bound) of the sequence $\{u_n\}$ implies that

$$u_n \leqslant M \ \forall \ n \in N$$

and $u_n > M - \epsilon$ for at least one value of n.

(iii) m is the lower bound (or greatest lower bound) of the sequence $\{u_n\}$ implies that

$$u_n \geqslant m \ \forall n \in N$$

and $u_n < m+\epsilon$ for at least one value of n.

(iv) $\{u_n\}$ is monotonic increasing $\Leftrightarrow u_n \geqslant u_m \ \forall n \geqslant m$.

(v) $\{u_n\}$ is monotonic decreasing $\Leftrightarrow u_n \leqslant u_m \ \forall n \geqslant m$.

(vi) $\{u_n\}$ is said to have the limit l iff given $\epsilon > 0$ there exists $m(\epsilon) \in N$ such that $n \geqslant m \Rightarrow |u_n - l| < \epsilon$.

4. (i) $\{u_n\}$ converges to l (or $u_n \to l$ as $n \to \infty$) iff given $\epsilon > 0$ there exists $m \in N$ such that $|u_n - l| < \epsilon$ whenever $n \geqslant m$.

(ii) $\{u_n\}$ is convergent iff given $\epsilon > 0$ there exists $\nu \in N$ such that $|u_{n+p} - u_n| < \epsilon \ \forall n \geqslant \nu$ and $p \in N$.

(iii) If $\lim\limits_{n\to\infty} u_n = l$, $\lim\limits_{n\to\infty} v_n = l \Rightarrow \lim\limits_{n\to\infty} (u_n + v_n) = l + l'$,

$$\lim_{n\to\infty} (u_n v_n) = ll', \quad \lim_{n\to\infty} \frac{u_n}{v_n} = \frac{l}{l'}, \text{ when } l' \neq 0.$$

(iv) Cauchy's First Theorem on Limits. If $a_n \to l$, as $n \to \infty$, then $\dfrac{a_1 + a_2 + a_3 + \ldots + a_n}{n} \to l$, as $n \to \infty$.

(v) $2 < \lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n < 3$, and $\lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n = e$.

(vi) $0 < \lim\limits_{n\to\infty} \left(1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n} - \log n\right) < 1$.

5. (i) Σu_n is convergent iff $\lim\limits_{n\to\infty} S_n$ is a definite number where $S_n = u_1 + u_2 + u_3 + \ldots + u_n$.

(ii) If Σu_n is convergent then $\lim\limits_{n\to\infty} u_n = 0$.

(iii) If Σu_n is a convergent series of positive decreasing terms then $\lim\limits_{n\to\infty} (n u_n) = 0$.

6. Auxiliary series for determining convergence—

(i) $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}+\ldots$ i.e. $\sum\frac{1}{n^p}$ is convergent if $p>1$ and divergent if $p\leqslant 1$.

(ii) $\sum_{n=2}^{\infty}\frac{1}{n(\log n)^p}$ is convergent if $p>1$ and divergent if $p\leqslant 1$.

(iii) $a+ar+ar^2+\ldots+ar^{n-1}+\ldots$ is convergent if $|r|<1$.

7. Tests for convergence of series of positive terms—

(a) (i) *Comparison Test.* $\sum u_n$ and $\sum v_n$, both of positive terms, converge or diverge together if $\lim_{n\to\infty}\frac{u_n}{v_n}=$ finite non-zero quantity.

(ii) *Second Comparison Test.* $\sum u_n$ of positive terms converges or diverges according as $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}>$ or $<\lim_{n\to\infty}\frac{v_n}{v_{n+1}}$ provided $\sum v_n$ is convergent or divergent.

(b) *Cauchy's Root Test.* $\sum u_n$ of positive terms is convergent or divergent according as $\lim_{n\to\infty} u_n{}^{1/n}<1$ or >1.

(c) (i) *(D' Alembert's) Ratio Test.* $\sum u_n$ of positive terms is convergent or divergent according as $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}>1$ or <1 (i.e. $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}<1$ or >1).

(ii) *Raabe's Test.* $\sum u_n$ of positive terms is convergent or divergent according as $\lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)\right\}>1$ or <1.

(iii) *DeMorgan and Bertrand's Test.* $\sum u_n$ of positive terms is convergent or divergent according as

$$\lim_{n\to\infty}\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right]>1 \text{ or } <1.$$

(iv) *Gauss' Test.* If $\sum u_n$ is a series of positive terms, for which $\frac{u_n}{u_{n+1}}=1+\frac{\alpha}{n}+\frac{b_n}{n^p}$, $p>1$ and $|b_n|\to$ definite number then the series is convergent if $\alpha > 1$ and divergent of $\alpha \leqslant 1$.

(d) (i) *Logarithmic Test.* Σu_n of positive terms is convergent or divergent respectively if $\lim\limits_{n\to\infty} \left(n \log \dfrac{u_n}{u_{n+1}}\right) > 1$ or < 1.

(ii) *Higher Logarithmic Test.* Σu_n of positive terms is convergent or divergent if $\lim\limits_{n\to\infty} \left[\left(n \log \dfrac{u_n}{u_{n+1}} - 1\right) \log n\right]$ > 1 or < 1 respectively.

(e) *Cauchy's Condensation Test.* $\Sigma f(n)$, where $f(n)$ is positive and decreasing, and $\Sigma a^n f(a^n)$ where $a > 1$, converge or diverge together.

8. Convergence of series of positive and negative terms—

(a) *Leibnitz's Rule.* Σu_n, a series of alternately positive and negative terms, is convergent if $\lim\limits_{n\to\infty} u_n = 0$ and $u_{n+1} \leqslant u_n$ for all $n \in N$.

(b) Σu_n of terms of arbitrary signs, is convergent if $\Sigma \mid u_n \mid$ is convergent.

(c) Σu_n of terms of arbitrary signs, is conditionally convergent if Σu_n is convergent but $\Sigma \mid u_n \mid$ is divergent.

9. Continuity and differentiability of functions—

(a) Verification of continuity—

(i) A function $f(x)$ is continuous at $x = a$ if $\lim\limits_{h\to 0} f(a+h) = \lim\limits_{h\to 0} f(a-h) = f(a)$.

(ii) A function $f(x)$ is continuous at $x = a$, the left-end point of the domain, if $\lim\limits_{h\to 0} f(a+h) = f(a)$.

(iii) A function $f(x)$ is continuous at $x = a$, the right-end point of the domain, if $\lim\limits_{h\to 0} f(a-h) = f(a)$.

(b) Verification of derivability—

(i) A function $f(x)$ is derivable at $x = a$ if

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{f(a-h) - f(a)}{-h}$$

$= \text{a definite number.}$

Some Hindi-English Equivalent Mathematical Words

English	*Hindi*
Addition law	योग नियम
Associative law	साहचर्य नियम
Archimedean law	आर्किमिडीय नियम
Absolute value	निरपेक्ष मान
Arbitrary number	स्वेच्छ संख्या
Algebraic function	बीजीय फलन
Alternating series	एकान्तर श्रेणी
Absolute convergence	निरपेक्ष अभिसरण
Bound	परिबंध
Bounded	परिबद्ध
Bonded sequence	परिवद्ध अनुक्रम
Closure law	निमीलन नियम
Commutative law	क्रमविनिमेय नियम
Co-efficient	गुणांक
Convergent	अभिसारी
Convergence	अभिसरण
Converges to the limit	सीमा की ओर अभिसारी
Closed internal	बन्द अन्तराल
Cauchy's general principle of convergence	कोशि के अभिसरण का सामान्य सिद्धान्त
Continuous function	संतत फलन
Continuity	सांतत्य
Continuous variable	संतत चर
Continuous property	सांतत्यक प्रगुण
Distributive law	वंटन नियम
Domain	प्रांत
Divergent	अपसारी
Divergence	अपसरण
Differentiability	अवकलनीयता
Differentiable	अवकलनीय
Differential	अवकल
Differentiable from the right	दायीं ओर से अवकलनीय
Discontinuity	असांतत्य
Derivable	अवकलज

English	*Hindi*
Even function	सम फलन
Function	फलन
Finite series	परिमित श्रेणी
Geometric series	गुणोत्तर श्रेणी
General term	सामान्य पद
Greatest lower bound	महत्तम निम्न परिबंध
Irrational number	अपरिमेय संख्या
Integer	पूर्णांक
Identity element	तत्समक अवयव
Inverse element	प्रतीप अवयव
Interval	अन्तराल
Infinite series	अनन्त श्रेणी
Inferior limit	निम्न सीमा
Lower bound	निम्न परिवंध
Least upper bound	न्यूनतम उपरि परिवंध
Limit	सीमा
Limit of a sequence	अनुक्रम की सीमा
Limit point	सीमा-विन्दु
Lefthand limit	बायीं सीमा
Monotony law	एकदिष्टता नियम
Modulus	मापांक
Monotonic sequence	एकदिष्ट अनुक्रम
Monotonic increasing sequence	एकदिष्ट बर्द्धमान अनुक्रम
Monotonic decreasing sequence	एकदिष्ट ह्रासमान अनुक्रम
Necessary condition	आवश्यक प्रतिबन्ध
Neighbourhood	सामीप्य
Norm	मानक
Odd function	विषम फलन
Open interval	खुला अन्तराल
Order relation	क्रम सम्बन्ध
Ordered field	क्रमित फील्ड
Oscillatory series	दोलायमान श्रेणी
Partial sum	आंशिक योग
Particular term	विशेष पद या निर्दिष्ट पद
Polynomial function	बहुपदी फलन
Range	परिसर
Ratio Test	अनुपात जाँच
Rational numbers	परिमेय संख्याएं

English.	*Hindi*
Real numbers	वास्तविक संख्याएं
Real sequence	वास्तविक अनुक्रम
Righthand limit	दायीं सीमा
Sequence	अनुक्रम
Series	श्रेणी
Sufficient condition	पर्याप्त प्रतिबन्ध
Superior limit	उच्च सीमा
Tends to the limit	सीमा की और प्रवृत्त होता है
Terms	पद
Transcendental function	अबोजीय फलन
Transitive law	संक्रमक नियम
Trichotomy	त्रिविभागी
Unbounded	परिबद्ध नहीं या अपरिवद्ध
Unique limit	अद्वितीय सीमा
Upper bound	उपरि परिबंध

□

CHAPTER
1

LEAST UPPER BOUND AND GREATEST LOWER BOUND IN *R*

1.1. Introduction

The underlying set of our analysis is the set R of real numbers. The students are already familiar with the set N of natural numbers, the set Z of integers, the set Q of rationals and the set P of irrationals intuitatively. Remember

$$N = \{1, 2, 3, \ldots\}$$

$$Z = \{\ldots, -3, -2, -1, 0, 1\ 2, 3, \ldots\}$$

$$Q = \left\{x \mid x = \frac{p}{q} \text{ where } p \in Z,\ q \in Z,\ q \neq 0 \text{ and } p, q \text{ are in the lowest order}\right\}$$

$$= \left\{0, 1, 2, \ldots, \frac{1}{2}, \frac{3}{4}, \frac{5}{4}, \ldots, -1, -2, \ldots, -\frac{1}{2}, -\frac{3}{4}, \frac{-5}{4}, \ldots\right\}$$

$$P = \{x \mid x = a^{\frac{1}{n}} \text{ or } \pi \text{ or } e$$

where $a \in Q$ and a is not equal to n^{th} power of a rational}
and $R = Q \cup P,\ N \subset Z \subset Q \subset R,\ Q \cap P = \varphi$.

Note. π and e are also know as transcendental numbers. They are special irrational numbers.

1.2. Structure of *R*

The various algebraic laws that hold in R for two binary operations addition ($+$) and multiplication ($\times$) and an order relation $>$, are as follows—

Addition (+) Laws

(A_1) *Closure Law.* $a, b \in R \Rightarrow a + b \in R$.

(A_2) *Associative Law.* $a, b, c \in R$

$$\Rightarrow (a + b) + c = a + (b + c)$$

(A_3) *Commutative Law.* $a, b \in R \Rightarrow a + b = b + a$

(A_4) *Existence of Identity Element of Addition (or Additive Identity).* There exists a number $0 \in R$, called zero, such that

$$a + 0 = 0 + a = a, \ \forall \ a \in R.$$

(A_5) *Existence of Inverse Elements of Addition (or Additive Inverses).* For each $a \in R$ there exists a corresponding element $b \in R$ such that $a + b = b + a = 0$.

b is called additive inverse or negative of a and it is written as $-a$.

Multiplication ($\times$) Laws.

(M_1) *Closure Law.* $a, b \in R \Rightarrow a \times b \in R$

(M_2) *Association Law.* $a, b, c \in R$

$$\Rightarrow a \times (b \times c) = (a \times b) \times c$$

(M_3) *Commutative Law.* $a, b \in R \Rightarrow a \times b = b \times a$

(M_4) *Existence of Identity Element of Multiplication (or Multiplicative Identity).* There exists a number $1 \in R$, called unity, such that $a \times 1 = 1 \times a = a \ \forall \ a \in R$

(M_5) *Existence of Inverse Elements of Multiplication (or Multiplicative Inverses).* For each $a \in R$ there exists a corresponding $b \in R$ such that $a \times b = b \times a = 1$.

b is called multiplicative inverse (also called reciprocal) of a and it is denoted as a^{-1} or $\frac{1}{a}$.

Distributive Laws for ($+$) and ($\times$)

(D) $a, b, c \in R \Rightarrow a \times (b + c) = a \times b + a \times c$.

Laws of Order Relation $>$ (greater than)

(O_1) *Trichotomy Law.* If $a, b \in R$ then one and only one among the following holds—

$$a > b \ \text{ or } \ a = b \ \text{ or } \ b > a.$$

(O_2) *Transitivity Law.* For $a, b, c \in R$ if $a > b$ and $b > c$ then

$$a > c.$$

(O_3) *Monotony Law for Addition.* For $a, b, c \in R$ if $a > b$ then $a + c > b + c$.

(O_4) *Monotony Law for Multiplication.* For $a, b, c \in R$ if $a > b$ and $c > 0$ then $a \times c > b \times c$.

(O_5) *Archimedean Law.* If $a, b \in R$ and $a > 0$ then there exists a natural number n such that $na > b$.

[**Note.** The order relation laws can be expressed in terms of $<$ (less than) for $a > b$ and $b < a$ carry the same meaning.]

In view of the algebraic laws $(A_1) - (A_5)$, $(M_1) - (M_5)$ and (D) that hold for R, algebraic structure of R from the point of views of Abstract Algebra is that of a field i.e. R is a field for the operations $+$ and $\times$.

As R also obeys the laws $(O_1) - (O_4)$ for order relation $>$, we say R is an ordered field. The above mentioned algebraic laws and laws for order also hold for the set Q of rationals.

As such Q is also an ordered field.

As $0 \in R$, for every $a \in R$ we have any one of the following, according to the law (O_1)—

$$a > 0 \quad \text{or} \quad a = 0 \quad \text{or} \quad 0 > a.$$

If $a > 0$, a is called a *positive* real number and if $0 > a$, a is called a *negative* real number. The set of positive real numbers will be denoted by R^+ and that of negative real numbers by R^-. Also if $x \in R^+$, $y \in R^{-1}$ then $x > y$.

1.3. Absolute Value or Modulus or Norm of a Real Number

The absolute value or norm of a real number x is denoted by $| x |$ and it is defined as follows—

$$\begin{aligned} | x | &= x \text{ when } x \geqslant 0 \\ &\quad - x \text{ when } 0 > x \end{aligned}$$

This can also be defined as $| x |$ = maximum of x and $-x$.

Absolute value of $a - b$ i.e. $a + (-b)$ is denoted by $| a - b |$.

By definition of absolute value,

$$\begin{aligned} | a-b | &= a - b \text{ if } a > b \\ &\quad b - a \text{ if } b > a \end{aligned}$$

$\therefore \quad |a-b| < \epsilon \Leftrightarrow a - b < \epsilon$ if $a > b$

or $\qquad b - a < \epsilon$ if $b > a$

Again $a - b < \epsilon \Leftrightarrow a < b + \epsilon$

$\qquad b - a < \epsilon \Leftrightarrow b - \epsilon < a$

Thus $$\boxed{|a-b| < \epsilon \Leftrightarrow b - \epsilon < a < b + \epsilon} \quad \ldots \text{(A)}$$

Again if $|a-b| < k\epsilon$, where k is a positive real number and ϵ is an arbitrary positive real number, then $a = b$.

For if $a \neq b$ taking $\epsilon = \frac{1}{2k}|a-b| > 0$, as ϵ is arbitrary,

we get $|a-b| < k\epsilon \Rightarrow |a-b| < k.\frac{1}{2k}|a-b|$

$\qquad \Rightarrow |a-b| < \frac{1}{2}|a-b|$

which is absurd for $|a-b|$ is positive.

Hence $a \neq b$ is impossible. So $a = b$.

Thus

$$\boxed{k \in R^+ \text{ and } |a-b| < k.\epsilon \; \forall \; \epsilon > 0 \Rightarrow a = b} \quad \ldots \text{(B)}$$

Other useful properties of the absolute value are given below—

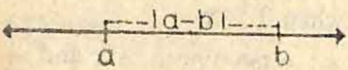
(i) $|a-b| = |b-a|$, (symmetry)

(ii) $|a-b| \leqslant |a-c| + |c-b|$, (triangle inequality)

Replacing b by $-b$ and c by 0 (ii) becomes

$$|a+b| \leqslant |a| + |b|.$$

Note. $|a-b|$ is also called the distance between the real numbers a and b. If the distance between a and b is denoted by $d(a, b)$ then



$$d(a, b) = |a-b|.$$

1.4. Bounds of a Subset in *R*

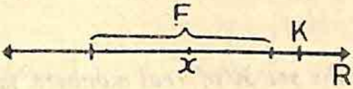
A real number K is called an *upper bound* of the subset F of R if K is greater than or equal to each element of F, Symbolically,

$$\boxed{x \leqslant K \ \forall \ x \in F \Leftrightarrow K \text{ is an upper bound.}}$$

It is clear that if K is an upper bound then $K+p$, where p is any positive real number, is also an upper bound.

If an upper bound of the subset F exists then we say that F is *bounded above*.

Pictorially,



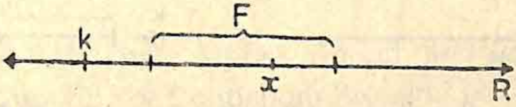
A real number k is called a *lower bound* of the subset F of R if every element of F is greater than or equal to k. Symbolically,

$$\boxed{x \geqslant k \ \forall \ x \in F \Leftrightarrow k \text{ is a lower bound.}}$$

It is clear that if k is a lower bound then $k-p$, where p is any positive real number, is also a lower bound.

If a lower bound of the subset F exists then we say that F is *bounded below*.

Pictorially,



A subset F of R is called *bounded* if it has an upper bound as well as a lower bound. In other words, the subset F of R is bounded if two real numbers k and K exist such that every element of F is greater than or equal to k and less than or equal to K. Symbolically,

$$\boxed{k \leqslant x \leqslant K \ \forall \ x \in F \Leftrightarrow F \text{ is bounded.}}$$

Clearly an upper bound is greater than or equal to every lower bound.

Example. Let $F = \left\{1, \dfrac{1}{2}, \dfrac{1}{2^2}, \ldots\right\}$. Here $F \subset R$.

Clearly $x \leqslant 1 \ \forall \ x \in F$. So 1 is an upper bound and 2, 3, 2·5 etc. are all upper bounds. Thus F is bounded above.

Again $x \geqslant 0 \ \forall \ x \in F$. So 0 is a lower bound. $-1, -2, -2{\cdot}3$ etc. are all lower bounds of F. So F is bounded below.

Thus $0 \leqslant x \leqslant 1 \ \forall \ x \in F$. So F is a bounded set.

1.5. Theorem

A subset F of the set R of real numbers is bounded if and only if there exists a number $l \in R$ such that

$$|x| \leqslant l \ \forall \ x \in F.$$

Proof. Let F be a bounded subset of R.

We have to prove that there exists a number $l \in R$ such that

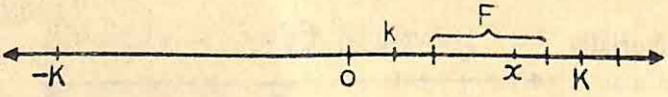
$$|x| \leqslant l \ \forall \ x \in F.$$

As F is bounded, by definition, there exists two real numbers k and K such that

$$k \leqslant x \leqslant K \ \forall \ x \in F. \qquad \ldots \quad (1)$$

If both k and K are positive then due to (1),

$$-K \leqslant x \leqslant K$$

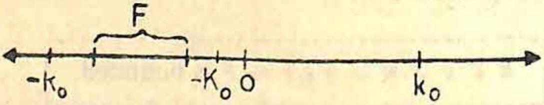


or $\quad 0 - K \leqslant x \leqslant 0 + K; \quad \therefore \quad |x| \leqslant K,$

[according to (*A*) of § 1.3]

If both k and K are negative then let $k = -k_0$, $K = -K_0$ where k_0, K_0 are positive.

Clearly due to (1), $-k_0 \leqslant x \leqslant k_0$

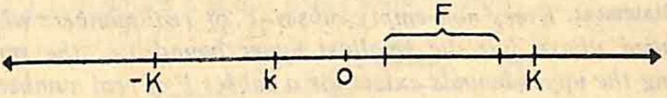


or $\quad 0 - k_0 \leqslant x \leqslant 0 + k_0; \quad \therefore \quad |x| \leqslant k_0,$

[according to (*A*) of § 1.3]

If k is negative and K is positive then let $k = -k_0$ where k_0 is positive.

Clearly due to (1), $-K \leqslant x \leqslant K$



or $0 - K \leqslant x \leqslant 0 + K$; $\therefore$ $|x| \leqslant K$, [according to (A) of § 1.3]

Thus in all cases $|x| \leqslant l \ \forall \ x \in F$ where $l =$ the greatest among K and k_0.

Next let F be a subset of R such that

$$|x| \leqslant l \ \forall \ x \in F \quad \text{i.e.} \quad |x - 0| \leqslant l \ \forall \ x \in F.$$

We have to prove that F is bounded.

By (A) of § 1.3, $0 - l \leqslant x \leqslant 0 + l$

i.e. $-l \leqslant x \leqslant l \ \forall \ x \in F.$

$\therefore$ F is bounded.

Thus we get

$$\boxed{F \text{ is bounded} \Leftrightarrow |x| \leqslant l \ \forall \ x \in F.}$$

1.6. Axioms of Least Upper Bound and Greatest Lower Bound (or Continuum Property or Completeness Property)

We know that if a subset F of real number set R has an upper bound K then F has many upper bounds, because

$$x \leqslant K \ \forall \ x \in F \Rightarrow x \leqslant K + p \ \forall \ x \in F$$

where p is any non-negative real number.

Similarly if F has a least upper bound k then F has many lower bounds because

$$x \geqslant k \ \forall \ x \in F \Rightarrow x \geqslant k - p \ \forall \ x \in F$$

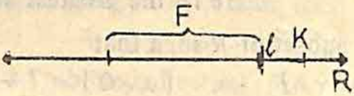
where p is any non-negative real number.

The following axiom states the most fundamental property for the analysis of the real number set R—

Axiom of Least Upper Bound

Statement. *Every non-empty subset F of real numbers which is bounded above has the smallest upper bound i.e. the smallest among the upper bounds exists for a subset F of real numbers.*

The smallest upper bound of a set F is called the *least upper bound (l.u.b.)* or *supremum* of F. If l is the supremum of F we write $l = \sup F$ or $\sup\limits_{x \in F} x$ and $l \leqslant K$ where K is any other upper bound. The least upper bound or supremum of a set F should not be confused with the *maximum* of F (i.e. the largest element of F). There are subsets of R for which supremum and maximum are equal. If F has a maximum M then clearly M is also the least upper bound. But we can have a subset of R for which the least upper bound exists but the maximum does not exist.



Example. (i) Let $F = \{1, 2, 3, 4, 5\}$; clearly $F \subset R$ and the maximum of $F = 5$. By definition of upper bound, K is an upper bound if $5 \leqslant K$. Hence the least upper bound of F = smallest value of $K = 5$.

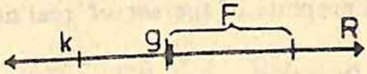
$\therefore$ Here the maximum of F = the least upper bound.

(ii) Let $F = \{x \mid 1 \leqslant x < 5, x \in R\}$. By definition of upper bound, K is an upper bound if $5 \leqslant K$. So the least upper bound is 5. But F does not have the maximum element (remember $5 \notin F$).

1.7. Axiom of Greatest Lower Bound

Statement. *Every non-empty subset F of real numbers which is bounded below has the largest lower bound* i.e. the largest among the lower bounds exists for a subset F of real numbers.

The largest among lower bounds of a subset F of real numbers is called the *greatest lower bound* ($g.l.b$) or infimum of F. If g is the infimum of F we write $g = \inf F$ or $\inf\limits_{x \in F} x$ and $g \geqslant k$ where k is any other lower bound.



The greatest lower bound or infimum of a set F is not always the same as the *minimum* of F (i.e. the smallest element of F). If F has a minimum m then clearly m is also the greatest lower bound. But we can have a subset of R for which the greatest lower bound exists but the minimum does not exist.

Example. (i) Let $F = \{x \mid 1 \leqslant x < 5, x \in R\}$.

By the definition of a lower bound, k is a lower bound of F if $k \leqslant 1$. So the greatest lower bound is 1. Also 1 is the minimum of F.

$\therefore$ Here the minimum of $F =$ the greatest lower bound.

(ii) Let $F = \{x \mid x > 1, x \in R\}$.

By the definition of a lower bound, k is a lower bound if $1 \geqslant k$. So the greatest lower bound is 1. But F does not have the minimum element.

If a set F does not have an upper bound we say F is *unbounded above* and we show it by writing $\sup F = +\infty$.

If F does not have a lower bound we say F is *unbounded below* and we show it by writing $\inf F = -\infty$.

1.8. Some Properties of Supremum and Infimum

Property 1. Let A be a non-empty bounded subset of R and λ is a non-negative real number.

Then (i) $\sup \lambda A = \lambda \sup A$.

(ii) $\inf \lambda A = \lambda \inf A$.

Proof. (i) Let $\sup A = l$. The existence of l is ensured by the fact that A is bounded above.

By definition of supremum, $x \leqslant l \;\forall\; x \in A$ .. (1)
and l is the smallest among such numbers.

If $\lambda > 0$, $\lambda x \leqslant \lambda l \;\forall\; x \in A$, using (1).

$\therefore$ λl is an upper bound of the set $\lambda A = \{\lambda x \mid x \in A\}$.

By continuum property of the set of real numbers, λA has the supremum.

Let $\sup \lambda A = l'$. If we can prove $\lambda l = l'$ our problem is solved.

As l' is the supremum and λl is an upper bound of λA, we get $\lambda l \geqslant l'$.

Again as l' is the supremum of λA, we get $\lambda x \leqslant l'$;

$$\therefore \quad x \leqslant \frac{l'}{\lambda} \;\forall\; x \in A$$

$\therefore$ $\dfrac{l'}{\lambda}$ is an upper bound of A. But l is the least upper bound of A. Hence $l \leqslant \dfrac{l'}{\lambda}$; so $\lambda l \leqslant l'$.

Thus we get $\lambda l \geqslant l'$ as well as $\lambda l \leqslant l'$. Hence we conclude $\lambda l = l'$.

$\therefore$ $\sup \lambda A = l' = \lambda l = \lambda \sup A$.

If $\lambda = 0$, the set $\lambda A = \{0x \mid x \in A\} = \{0\}$.

$\therefore$ $\sup \lambda A = \sup \{0\} = 0 = \lambda \sup A$.

[**Note.** This result holds even when A is only bounded above.]

(ii) Let $\inf A = g$. The existence of g is ensured by the fact that A is bounded below.

By definition of infimum, $x \geqslant g \;\forall\; x \in A$ .. (1)
and g is the greatest among such numbers.

As $\lambda > 0$, $\lambda x \geqslant \lambda g \;\forall\; x \in A$, using (1).

$\therefore$ λg is a lower bound of the set $\lambda A = \{\lambda x \mid x \in A\}$.

By continuum property of the set of real numbers, λA has the infimum.

Let $\inf \lambda A = g'$. We have to prove that $\lambda g = g'$.

As g' is the infimum and λg is a lower bound of λA, we get $g' \geqslant \lambda g$.

Again as g' is the infimum of λA, we get $\lambda x \geqslant g'$;

$$\therefore \quad x \geqslant \frac{g'}{\lambda} \ \forall \ x \in A.$$

$\therefore \quad \frac{g'}{\lambda}$ is a lower bound of A. But g is the greatest lower bound of A. Hence $g \geqslant \frac{g'}{\lambda}$; so $\lambda g \geqslant g'$.

Thus we get $g' \geqslant \lambda g$ as well as $\lambda g \geqslant g'$. Hence we conclude $\lambda g = g'$.

$\therefore \quad \inf \lambda A = g' = \lambda g = \lambda \inf A$.

If $\lambda = 0$, the set $\lambda A = \{0 \, x \mid x \in A\} = \{0\}$

$\therefore \quad \inf \lambda A = \inf \{0\} = 0 = \lambda \inf A$.

[**Note.** This result holds even when A is only bounded below.]

Property 2. If A is a bounded subset of real numbers and $\lambda \leqslant 0$ is a real number then

(i) $\sup \lambda A = \lambda \inf A$

(ii) $\inf \lambda A = \lambda \sup A$.

Proof. (i) Let $\inf A = g$.

By definition of infimum, $x \geqslant g \ \forall \ x \in A$ .. (1)
and g is the largest among such numbers.

If $\lambda < 0$, $\lambda x \leqslant \lambda g \ \forall \ x \in A$, using (1)

$\therefore \quad \lambda g$ is an upper bound of the set $\lambda A = \{\lambda x \mid x \in A\}$.

By continuum property of the set of real numbers, λA has the supremum. Let $\sup \lambda A = l$. We have to prove that $\lambda g = l$.

As l is the least upper bound (supremum) of λA, $l \leqslant \lambda g$;

also $\quad \lambda x \leqslant l$ i.e. $x \geqslant \frac{l}{\lambda} \ \forall \ x \in A$, $(\because \ \lambda < 0)$

$\therefore \quad \frac{l}{\lambda}$ is a lower bound of A. But g is the greatest lower bound of A. So $\frac{l}{\lambda} \leqslant g$ i.e. $l \geqslant \lambda g$, $(\because \ \lambda < 0)$.

Thus we get $l \leqslant \lambda g$ as well as $l \geqslant \lambda g$.

$\therefore \ l = \lambda g$ i.e. $\sup \lambda A = \lambda \inf A$.

If $\lambda = 0$, the set $\lambda A = \{0\,x \mid x \in A\} = \{0\}$

$\therefore \ \sup \lambda A = \sup \{0\} = 0 = \lambda \inf A$.

(ii) Let $\sup A = l$.

By definition of supremum, $x \leqslant l \ \forall \ x \in A$ .. (1)
and l is the smallest among such numbers.

If $\lambda < 0$, $\lambda x \geqslant \lambda l \ \forall \ x \in A$, using (1)

$\therefore \ \lambda l$ is a lower bound of the set $\lambda A = \{\lambda x \mid x \in A\}$.

By continuum property of the set of real numbers λA has the infimum. Let $\inf \lambda A = g$. We have to prove that

$$g = \lambda l.$$

As g is the infimum of λA, $g \geqslant \lambda l$ and also $\lambda x \geqslant g$

i.e. $$x \leqslant \frac{g}{\lambda} \ \forall \ x \in A, \ (\because \ \lambda < 0)$$

$\therefore \ \frac{g}{\lambda}$ is an upper bound of A. So $l \leqslant \frac{g}{\lambda}$

i.e. $$\lambda l \geqslant g, \ (\because \ \lambda < 0)$$

Thus $g \geqslant \lambda l$ as well as $\lambda l \geqslant g$.

So $g = \lambda l$ i.e. $\inf \lambda A = \lambda \sup A$.

If $\lambda = 0$, $\lambda A = \{0\,x \mid x \in A\} = \{0\}$

$\therefore \ \inf \lambda A = \inf \{0\} = 0 = \lambda. \sup A$.

Cor 1. Taking $\lambda = -1$ in property 2 we get the following relations between the supremum and the infimum of the set A—

(i) $\sup(-A) = -\inf A$.

(ii) $\inf(-A) = -\sup A$.

Property 3. If A and B are two bounded subsets of R and $A + B$ is a subset of R defined by

$$A + B = \{x \mid x = a + b, \ a \in A, \ b \in B\}$$

where $a + b$ denotes ordinary addition of real numbers a and b

then

(i) $\sup(A+B) = \sup A + \sup B$

(ii) $\inf(A+B) = \inf A + \inf B$.

Proof. (i) Let $\sup A = l$, $\sup B = l'$.

$$\therefore \quad a \leqslant l, \; b \leqslant l' \; \forall \, a \in A, \; b \in B.$$

$$\therefore \quad a + b \leqslant l + l'.$$

Thus the set of real numbers of the form $a+b$ i.e. the set $A+B$ has an upper bound $l+l'$.

$\therefore$ By continuum property of R, $A+B$ has the least upper bound (supremum). Let it be λ. We have to prove that

$$\lambda = l + l'.$$

By definition of supremum, $\lambda \leqslant l + l'$.

Next, as l is the smallest among upper bounds of A, $l - \epsilon$ is not an upper bound of $A \; \forall$ real $\epsilon > 0$.

i.e., $a \leqslant l - \epsilon$ is not true for all $a \in A$

i.e., there exists an element $a_1 \in A$ such that $a_1 > l - \epsilon$.

Similarly, there exists an element $b_1 \in B$ such that

$$b_1 > l' - \epsilon.$$

$\therefore \quad a_1 + b_1 > l + l' - 2\epsilon.$

But $a_1 + b_1 \in A + B$ whose supremum is λ.

So $$a_1 + b_1 \leqslant \lambda.$$

$$\therefore \quad \lambda \geqslant a_1 + b_1 > l + l' - 2\epsilon,$$

i.e. $$\lambda > l + l' - 2\epsilon \; \forall \; \epsilon > 0.$$

$\therefore \quad \lambda \not< l + l'$; so $\lambda \geqslant l + l'$.

Thus we get $\lambda \leqslant l + l'$ as well as $\lambda \geqslant l + l'$.

$\therefore \quad \lambda = l + l'$ i.e. $\sup(A+B) = \sup A + \sup B$.

[**Note.** This holds even when A and B are bounded above only.]

(ii) Let $\inf A = g$, $\inf B = g'$.

$\therefore \quad a \geqslant g, \; b \geqslant g' \; \forall \, a \in A, \; b \in B.$

$\therefore \quad a + b \geqslant g + g'.$

Thus the set of real numbers of the form $a+b$, i.e. the set $A+B$ has a lower bound.

$\therefore$ By continuum property of R, $A+B$ has the greatest lower bound (infimum). Let it be μ. We have to prove that

$$\mu = g + g'.$$

By definition of infimum, $\mu \geqslant g + g'$.

Now, as g is the largest among lower bounds of A, $g + \epsilon$ is not a lower bound of A $\forall$ real $\epsilon > 0$.

i.e. $a \geqslant g + \epsilon$ is not true for all $a \in A$

i.e. $\exists$ (there exists) an element $a_1 \in A$ such that $a_1 < g + \epsilon$.

Similarly, there exists an element $b_1 \in B$ such that

$$b_1 < g' + \epsilon.$$

$\therefore \quad a_1 + b_1 < g + g' + 2\epsilon.$

But $a_1 + b_1 \in A + B$ whose infimum is μ.

So $$a_1 + b_1 \geqslant \mu.$$

$\therefore \quad g + g' + 2\epsilon > a_1 + b_1 \geqslant \mu,$

i.e. $g + g' + 2\epsilon > \mu \;\forall\; \epsilon > 0.$

$\therefore \quad g + g' \nless \mu;$ so $g + g' \geqslant \mu.$

Thus we get $\mu \geqslant g + g'$ as well as $\mu \leqslant g + g'$.

$\therefore \quad \mu = g + g'$ i.e. $\inf(A+B) = \inf A + \inf B$.

[**Note.** This holds even when A and B are bounded below only.]

1.9. Intervals in *R*

In order to analyse the real number set R it is essential to associate subsets of R with real numbers. This purpose is served by taking subsets of R in a special way as given below—

Open and Closed Intervals

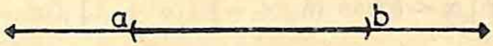
An open interval of real numbers is a subset of R defined as $\{x \mid a < x < b\}$ i.e. it is the set of all real numbers that are greater than a but less than b. This open interval is denoted by (a, b) or $]a, b[$. Thus $(a, b) = \{x \mid a < x < b, x \in R\}$

and $$y \in (a, b) \Rightarrow a < y < b.$$

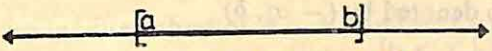
A closed interval of real numbers is a subset of R defined as $\{x \mid a \leqslant x \leqslant b\}$ i.e. it is the set of all real numbers that are equal to or greater than a but equal to or less than b. This closed interval is denoted by $[a, b]$. Thus

$$[a, b] = \{x \mid a \leqslant x \leqslant b, x \in R\}$$

and
$$y \in [a, b] \Rightarrow a \leqslant y \leqslant b.$$



open interval (a,b)

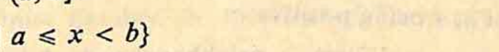


closed interval [a,b]

Clearly open and closed intervals are bounded sets in R. While in case of (a, b) there is no maximum or minimum element, in case of $[a, b]$ the maximum element is b and the minimum element is a.

There are other bounded intervals of R which are open at one end and closed at the other. These are also called semi-open or semi-closed intervals. For example—

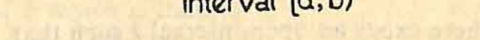
(i) $\{x \mid a < x \leqslant b\}$



interval(a,b]

This is closed at b but open at a and such intervals are denoted by $(a, b]$.

(ii) $\{x \mid a \leqslant x < b\}$

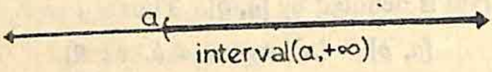


interval [a,b)

This is closed at a but open at b and such intervals are denoted by $[a, b)$.

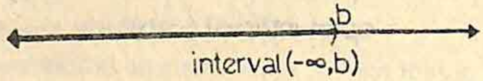
There are unbounded intervals of R. For example—

(1) $\{x \mid x > a\}$



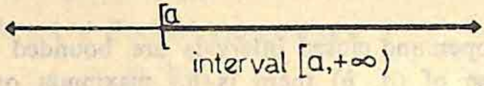
which is denoted by $(a, +\infty)$

(2) $\{x \mid x < b\}$



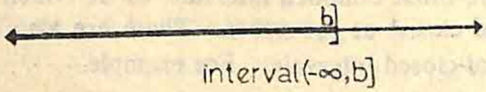
which is denoted by $(-\infty, b)$

(3) $\{x \mid x \geqslant a\}$



which is denoted by $[a, +\infty)$

(4) $\{x \mid x \leqslant b\}$



which is denoted by $(-\infty, b]$.

1.10. Neighbourhoods or Intervals Associated to a Real Number

For the real number a, the open interval $(a - \epsilon, a + \epsilon)$ i.e. the set $\{x \mid a - \epsilon < x < a + \epsilon\}$ is called an ϵ-neighbourhood of a, ϵ being positive.

In a more general form, a neighbourhood of a real number a is a subset T of R if (i) there exists $\epsilon > 0$ such that

$$(a - \epsilon, a + \epsilon) \subseteq T.$$

or (ii) there exists an open interval I such that

$$a \in I \subseteq T.$$

Example. (1) ϵ-neighbourhood of 1 is $(1 - \epsilon, 1 + \epsilon)$

(2) The set R is a neighbourhood of each real number x.

1.11. Length of an Interval

The length of each of the intervals (a, b), $[a, b]$, $(a, b]$ and $[a, b)$ is $b - a$.

The length of each of the intervals

$$(a, +\infty), [a, +\infty), (-\infty, b) \text{ and } (-\infty, b]$$

is infinite (∞).

1.12. Limit Points of an Infinite Subset of *R*

Let F be an infinite subset of R. A number $l \in R$ is called a limit point of F if every ϵ-neighbourhood of l contains infinite number of members of F. In other words,

$$x \in (l - \epsilon, l + \epsilon) \text{ i.e. } l - \epsilon < x < l + \epsilon$$

for infinite number of values of x from the set F.

1.13. Fundamental Theorem on Limit Point

(Bolzano-Weierstrass Theorem on Limit Point)

(M U 1966 H, '68 H, '72 H, '75 H; P U '69 H; Bh U '75 H)

Statement. Every infinite bounded subset of R has at least one limit point.

Proof. Let F be an infinite subset of R and it be bounded. By continuum property of R, F has a supremum l and an infimum g.

Let us define a set P as follows—

$P = \{x \mid x$ exceeds at the most a finite number of members of $F\}$.

As infimum g exceeds none of the members of F (i.e. exceeds a finite number of members of F) we get $g \in P$. So P is non-empty.

Again no real number greater than or equal to the supremum l can belong to P, because then it will exceed all members of F which is infinite.

So P is bounded above by l.

$\therefore$ By continuum property, P has a supremum ξ (say).

We shall prove that ξ is a limit point of F.

Let us take any ϵ-neighbourhood $(\xi - \epsilon, \xi + \epsilon)$ of ξ where $\epsilon > 0$.

As ξ is the supremum of P, ξ is the least among upper bounds of P i.e. P has no upper bound less than ξ. So $\xi - \epsilon$, where $\epsilon > 0$, is not an upper bound of P. Thus, there exists at least one element $y \in P$ such that $y > \xi - \epsilon$.

As $y \in P$, y exceeds at the most a finite number of members of F, by definition of P.

But $y > \xi - \epsilon$. So $\xi - \epsilon$ exceeds at most a finite number of members of F.

Next, ξ is the supremum of P. So $\xi + \epsilon$ exceeds all the elements of F which is infinite in number.

Thus $\xi - \epsilon < x < \xi + \epsilon$ for infinite number of members x of F.

$\therefore$ By definition of limit point ξ is a limit point of F.

WORKED OUT EXAMPLES

1. Prove that the set R^+ (i.e. the set of positive real numbers) is bounded below but unbounded above.

[सिद्ध करें कि समुच्चय R^+ (अर्थात् धन वास्तविक संख्याओं का समुच्चय) निम्न-परिबद्ध है किन्तु उपरि-परिबद्ध नहीं ।]

Solution. Here $x \in R^+ \Rightarrow x > 0 \ \forall \ x \in R^+$.

So 0 is a lower bound of R^+. Thus R^+ is bounded below.

If possible, let l be an upper bound of R^+

i.e. $$x \leqslant l \ \forall \ x \in R^+. \quad \text{.. (1)}$$

As $1 \in R^+$, $1 \leqslant l$. Also $0 < 2$.

$\therefore$ $1 + 0 < l + 2$; or $0 < l + 1$. Hence $l + 1 \in R^+$.

$\therefore$ From (1), $\quad l + 1 \leqslant l$

which is a contradiction for all $l \in R$.

Hence our assumption that l is an upper bound of R^+ gives a contradiction. So R^+ has no upper bound i.e. R^+ is unbounded above.

2. Find the greatest lower bound of the set A where

$$A = \{x \mid x = 4 + n^2 + \frac{1}{n^2},\ n \in N\}.$$

[समुच्चय $A = \{x \mid x = 4 + n^2 + \frac{1}{n^2},\ n \in N\}$ का महत्तम निम्न परिबंध निकालें ।]

Solution. Here $x = 4 + n^2 + \frac{1}{n^2} = 6 + \left(n^2 + \frac{1}{n^2} - 2\right)$

$$= 6 + \left(n - \frac{1}{n}\right)^2 \text{ where } n \in N.$$

$\therefore \quad x \geqslant 6 \;\forall\; x \in A.$ Hence 6 is a lower bound.

By continuum property A has the greatest lower bound g.

$$\therefore \quad x \geqslant g \geqslant 6.$$

Let $$g = 6 + \epsilon.$$

$\therefore \quad x \geqslant 6 + \epsilon;$ or $x = \left(n + \frac{1}{n}\right)^2 + 2 \geqslant 6 + \epsilon \;\forall\; n \in N.$

Taking $n = 1 \in N$, $(1 + 1)^2 + 2 \geqslant 6 + \epsilon$ i.e. $0 \geqslant \epsilon$.

But $0 > \epsilon$ implies $g = 6 + \epsilon < 6$ which contradicts the fact $g \geqslant 6$.

$$\therefore \quad \epsilon = 0 \text{ i.e. } g = 6.$$

Hence the greatest lower bound of A is 6.

3. If $x > 0$, prove that given $y \in R$ there exists $n \in N$ such that $nx > y$. (Archimedean property)

[यदि $x > 0$, सिद्ध करें कि प्रदत्त $y \in R$ के लिए एक $n \in N$ प्राप्त होगा ताकि $nx > y$. (आरकिमेडीयन प्रगुण)

Solution. As $y \in R$, $y \leqslant 0$ or $y > 0$.

When $y \leqslant 0$, $nx > y$ is obvious, as $x > 0$.

When $y > 0$, if possible let $nx \leqslant y \;\forall\; n \in N$.

Thus the set $A = \{nx \mid n \in N\}$ is bounded above, y being an upper bound.

$\therefore$ By continuum property, A has supremum.

Let $\sup A = l$. So $nx \leqslant l \;\forall\; n \in N$.

$\therefore$ As $n + 1 \in N$, $(n + 1)\,x \leqslant l \;\forall\; n \in N$

or $$nx \leqslant l - x \;\forall\; n \in N.$$

$\therefore$ $l - x$ is also an upper bound of the set A.

As $x > 0$, $l - x < l$. But l is the smallest among upper bounds. So $l - x < l$ contradicts the meaning of supremum.

$\therefore$ Our assumption, $nx \leqslant y \;\forall\; n \in N$ was wrong.

$\therefore$ There exists at least one n such that $nx > y$.

Problem—Set 1

1. (i) Prove that the set of negative real numbers is bounded above but unbounded below.

[सिद्ध करें कि ऋण वास्तविक संख्याओं का समुच्चय उपरि-परिबद्ध है किन्तु निम्न-परिबद्ध नहीं ।]

(ii) Prove that the set R of all real numbers is an unbounded set.

[सिद्ध करें कि वास्तविक संख्याओं का समुच्चय R अपरिबद्ध है ।]

2. (i) Find the greatest lower bound of the set

$$\{n^2 + \frac{1}{n^2} + 1 \mid n \in N\}.$$

[समुच्चय $\{n^2 + \frac{1}{n^2} + 1 \mid n \in N\}$ का महत्तम निम्न-परिबंध निकालें ।]

(ii) Find the least upper bound of the set

$$\{3 - n + \frac{1}{n} \mid n \in N\}.$$

[समुच्चय $\{3 - n + \frac{1}{n} \mid n \in N$ का निम्नतम उपरि-परिबंध निकालें ।]

(iii) Prove that the greatest lower bound of the set

$\{x \mid x > 0\}$ is 0.

[सिद्ध करें कि समुच्चय $\{x \mid x > 0\}$ का महत्तम निम्न परिबंध 0 है ।]

(iv) If x and y are lower and upper bounds for a set $F \leqslant R$ such that $x, y \in F$ then prove that

$$\sup F = y \quad \text{and} \quad \inf F = x.$$

[यदि किसी समुच्चय $F \leqslant R$ के निम्न एवं उपरि परिबंध क्रमशः x और y हों एवं $x, y \in F$ तो सिद्ध करें कि

$$\sup F = y \quad \text{and} \quad \inf F = x.]$$

3. Prove that, if $x \in R$ there exists a unique $n \in Z$, the set of integers, such that

$$n \leqslant x < n + 1.$$

[यदि $x \in R$ तो सिद्ध करें कि एक अद्वितीय $n \in Z$ (पूर्णांकों का समुच्चय प्राप्त होगा ताकि $n \leqslant x \leqslant n + 1$.]

CHAPTER 2

REAL SEQUENCES

The concept of sequences in the real number set is a powerful tool for the analysis of R. In fact George Cantor (1845-1918) made the sequences of rational numbers as the basis for most of the fundamental concepts in real analysis.

2.1. Functions in R

A rule or correspondence f is called a real valued function if it associates each member x of the set A to an uniquely determined member y (called image of x) of B, a subset of R. It is denoted by $f : A \rightarrow B$ or $y = f(x)$, indicating that the real number y corresponds to $x \in A$ under the rule f. The set A is called domain of the function and the set $f(A) = \{y \mid y = f(x); x \in A\}$ i.e., the set of images of A is called the range of f. Also x is called an independent variable and y is called a dependent variable.

Features of a function $y = f(x)$ in R are as follows—

(i) y is uniquely determined for every value of x from the domain. The value of y corresponding to the value a of x is denoted by $f(a)$.

(ii) For two or more values of x, y can have the same value.

(iii) The image i.e. the value of y for a prescribed value of x is obtained by substituting the value of x in the equation connecting x, y to describe the rule f.

2.2. Some Classifications of Real Functions

(1) Polynomial Function.

A function $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n$ where $a_0, a_1, a_2, \dots, a_n \in R$ and are independent of $x \in R$, is called a polynomial function in x.

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(2) Rational Function.

A function $f(x)$ of the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$, $(\neq 0)$ are polynomial functions in x, is called a rational function.

In particular when $Q(x)$ is a constant i.e. independent of x then $f(x)$ reduces to a polynomial function. Thus every polynomial function is a rational function.

(3) Algebraic Functions.

The function $y = f(x)$ is an algebraic function of x, if y is the root of an equation of the nth degree in y whose coefficients are rationl functions of x

i.e. $y^n + R_1(x)y^{n-1} + \ldots + R_n(x) = 0.$

(4) Transcendental Functions.

All functions which are not algebraic are called transcendental; this is a wide class of functions and includes such well-known functions as $\sin x$, $\cos x$ and $\log x$, as well as many which are less familiar.

(5) Odd and Even Functions.

If a function $f(x)$ changes its sign as the sign of x is changed i.e. $f(x) = -f(-x)$ then that function is called an odd function of x.

Example. $f(x) = x + 5x^3$ is an odd function, because

$$f(-x) = -x + 5(-x)^3 = -(x + 5x^3) = -f(x).$$

If a function $f(x)$ does not change its sign as the sign of x is changed, then that function is called an *even function* of x.

Example. $f(x) = x^4 + 8x^2$ is an even function, because

$$f(x) = x^4 + 8x^2 = (-x)^4 + 8(-x)^2 = f(-x).$$

2.3. Sequence

(M. U. 1984)

A function u whose domain is the set of natural numbers N is called a *sequence*. Thus a real sequence u is a correspondence which associates each natural number to uniquely determined real number. The image of the natural number n is denoted by u_n or $u(n)$. It is customary to depict a sequence by listing the images of the natural numbers 1, 2, 3, ...in that order. Thus a sequence u is denoted by $u_1, u_2, u_3, \ldots, u_n, \ldots$ in the long hand and briefly it is written as $\{u_n\}$ or $< u_n >$ or (u_n).

The image of n i.e. u_n is called the nth term (or element) of the sequence.

Example. (i) $1, 3, 5, 7, \ldots, 2n-1, \ldots$ is a sequence for which $u_1 = 1$, $u_2 = 3$, $\ldots$, $u_n = 2n-1$, etc.

(ii) $-1, +1, -1, +1, \ldots$ is a sequence for which $u_1 = -1$, $u_2 = +1 = (-1)^2$, $\ldots$, $u_n = (-1)^n$, etc.

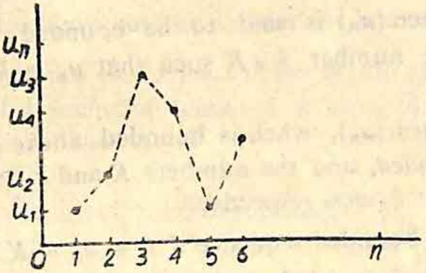
The set $\{x \mid x = u_n, n \in N\}$, of distinct terms of the sequence $\{u_n\}$, is called the *range* or *range-set* of the sequence $\{u_n\}$. The set of terms of the sequence $\{u_n\}$ is the set

$$u = \{u_1, u_2, u_3, \ldots, u_n, \ldots\}.$$

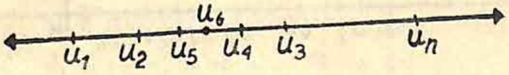
Example. (a) The range set of $\{u_n\}$, $u_n = 3n+1$ is the set $\{4, 7, 10, 13, \ldots\}$ which is a subset of R.

(b) The range set of the sequence in (ii) is $\{-1, 1\}$.

As a sequence is a function, the diagram of a sequence $\{u_n\}$ is shown as follows—



As the terms of a real sequence are real numbers, a sequence is also shown by points on the number line as follows—



[**Note.** From the definition of a sequence as a function from N to R, it is clear that the number of terms (distinct or equal) in a sequence $\{u_n\}$ is infinite. In the elementary level, students came across progressions which were also termed sequences and many of them were of finite number of terms. For example—

(i) $1, 3, 5, 7, \ldots, 25;$

(ii) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^{n-1}}.$

This must not lead to any confusion for it should be borne in mind that actually in writing the progression (i) or (ii) we have only indicated the first few terms of an infinite sequence.

In (i) the sequence is $\{2n-1\}$ and

in (ii) the sequence is $\left\{\frac{1}{2^{n-1}}\right\}$.]

In order to investigate the behaviour of a sequence, the following concepts are to be understood properly.

2.4. Bounded Sequences (P U 1967 H, '62 H)

Definitions. The sequence $\{u_n\}$ is said to be *bounded above* if there exists a finite number $K \in R$ such that

$$u_n \leqslant K, \text{ for all values of } n \in N.$$

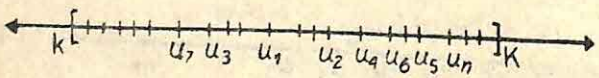
The sequence $\{u_n\}$ is said to be bounded below if there exists a finite number $k \in R$ such that $u_n \geqslant k$, for all values of $n \in N$.

The sequence $\{u_n\}$, whch is bounded above and below is said to be *bounded*, and the numbers K and k are called *rough upper* and *lower bounds* respectively.

$\therefore$ u_n is a bounded sequence if $k \leqslant u_n \leqslant K \; \forall \; n \in N$.

In other words $\{u_n\}$ is bounded if $u_n \in [k, K] \; \forall \; n \in N$.

Pictorially—



2.5. Least Upper and Greatest Lower Bounds

Definitions. If M is a number such that

(i) $u_n \leqslant M$, for all values of $n \in N$

and (ii) $u_n > M - \epsilon$, for at least one value of $n \in N$ where ϵ is an arbitrary positive number, however small, then M is called *the least upper bound* or simply *the upper bound* of the sequence $\{u_n\}$, which is bounded above.

Clearly the upper bound of $\{u_n\}$ is also a rough upper bound of $\{u_n\}$.

If m is a number such that

(i) $u_n \geqslant m$ for all values of n,

and (ii) $u_n < m + \epsilon$, for at least one value of $n \in N$ where ϵ is an arbitrary positive number, however small, then m is called the *greatest lower bound* or simply *the lower bound* of the sequence $\{u_n\}$, which is bounded below.

Clearly the lower bound of $\{u_n\}$ is also a rough lower bound of $\{u_n\}$.

Examples. (i) The sequence $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots, \frac{n+1}{n}, \ldots$ is bounded, as its upper bound is 2 and the lower bound is 1. A rough upper bound is 3 and a rough lower bound is 0.

(ii) The sequence $1, 2, 3, \ldots, n, \ldots$ is bounded below but it is not bounded above and hence it is not bounded.

[**Note.** The existence of the least upper bound and the greatest lower bound of a bounded sequence is ensured by the axiom of completeness for the set R.]

2.6. Monotonic or Monotone Sequences

(B U 1975; M U '65 H, '68 H)

If $u_{n+1} \geqslant u_n$ for all values of $n \in N$, then u_n is said to increase steadily and the sequence $\{u_n\}$ is said to be a monotonic increasing sequence.

Explicitly, for a monotonic increasing sequence $\{u_n\}$,

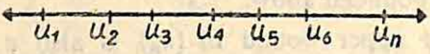
$$u_1 \leqslant u_2 \leqslant u_3 \leqslant \ldots \leqslant u_n \leqslant \ldots.$$

If $u_{n+1} > u_n$ for all values of $n \in N$, then the sequence $\{u_n\}$ is said to be strictly monotonic increasing.

Thus

a sequence $\{u_n\}$ is monotonic increasing $\Leftrightarrow u_n \geqslant u_m \,\forall\, n \geqslant m.$

Pictorially,



Obviously, for a m.i. (monotonic increasing) sequence $\{u_n\}$, $u_n \geqslant u_1 \,\forall\, n \in N$. So a m.i. sequence is bounded below, u_1 being a rough lower bound.

Examples. (i) The sequence defined by 1, 1, 3, 3, 5, 5, ... is monotonic increasing.

(ii) The sequence defined by 2, 4, 6, ..., $2n$, ... is a strictly monotonic increasing sequence.

If $u_{n+1} \leqslant u_n$ for all values of $n \in N$, then u_n is said to decrease steadily and the sequence $\{u_n\}$ is said to be a monotonic decreasing sequence.

Explicitly for a monotonic decreasing sequence,

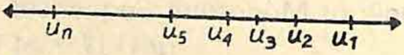
$$u_1 \geqslant u_2 \geqslant u_3 \geqslant \ldots . \geqslant u_n \geqslant \ldots .$$

$u_{n+1} < u_n$ for all values of $n \in N$, then the sequence $\{u_n\}$ d to be strictly monotonic decreasing.

Thus

sequence $\{u_n\}$ is monotonic decreasing $\Leftrightarrow u_n \leqslant u_m \,\forall\, n \geqslant m.$

Pictorially,



Obviously for a m.d. (monotonic decreasing) sequence $u_n \leqslant u_1 \,\forall\, n \in N$. So a m.d. sequence is bounded above, u_1 being a rough upper bound.

Examples. (i) The sequence defined by

$$-1, -1, -3, -3, -5, -5, \ldots .$$

is monotonic decreasing.

(ii) The sequence defined by

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$$

is a monotonic decreasing sequence in the stricter sense.

A sequence which is either monotonic increasing or monotonic decreasing is said to be *monotonic*.

2.7. Limit of a Sequence (M. U. 1984)

In case of a sequence which has infinite terms it is difficult to keep track of all the terms individually. So in order to analyse a sequence it is found convenient to describe the location of the terms in the real number set R by considering nearness of most of the terms to a prescribed real number. With this in view the concepts of limits and convergence are introduced for a sequence.

Limit— *The sequence* $\{u_n\}$ *is said to have the limit* l *if for a given positive number* ϵ *there exists a positive integer* m *such that* $|u_n - l| < \epsilon$ *for all integral values of* $n \geqslant m$.

Obviously m *depends on* ϵ in general.

If $\{u_n\}$ has the limit l we show it by writing $u_n \to$ $n \to \infty$ or $\lim\limits_{n \to \infty} u_n = l$.

In more compact symbolic forms—

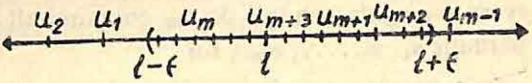
The sequence $\{u_n\}$ has the limit l if

(i) $\epsilon > 0 \ \exists \ m(\epsilon) \in N.\ n \geqslant m \Rightarrow |u_n - l| < \epsilon$.

i.e. $l - \epsilon < u_n < l + \epsilon$ [see (A) in section 1.3]

or (ii) $\epsilon > 0 \ \exists \ m(\epsilon) \in N.\ n \geqslant m \Rightarrow u_n \in (l - \epsilon,\ l + \epsilon)$.

Pictorially,



Thus l is a limit if in every ϵ-neighbourhood of l contains all the terms of the sequence $\{u_n\}$ except perhaps a few.

As ϵ is an arbitrary positive real number, it can be taken as small as we please and as such physically, l is the limit means that except a finite number of terms of the sequence, all the terms are situated very close to l on the number line R.

Note. (i) Some earlier authors use the word "however small" for ϵ which is superfluous as arbitrariness of ϵ includes that fact.

(ii) In the definition, m need not be taken an integer

$$\text{“}n \geqslant 5\tfrac{1}{8} \Rightarrow |\, u_n - l \,| < \epsilon\text{”}$$

can imply "$n \geqslant 6 \Rightarrow |\, u_n - l \,| < \epsilon$" because n is a natural number.

2.8. Limit Point of a Sequence, Limit Superior and Limit Inferior

Limit Point. A real number l is called the limit point of a sequence if every ϵ-neighbourhood of l contains an infinite number of terms of the set of the terms of the sequence.

Clearly if for a sequence $\{u_n\}$, $u_n = l$ for infinite values of n then l is a limit point of the sequence.

***Theorem 2.1.** (Bolzano-Weierstrass Theorem)

Every bounded sequence has a limit point.

roof. Let $\{u_n\}$ be a bounded sequence and u be the set of terms of the sequence i.e. $u = \{u_1, u_2, u_3, \ldots, u_n, \ldots\}$.

If $\{u_n\}$ has infinitely many distinct points then the result mmediately follows from Bolzano-Weierstrass theorem on limit point in chapter 1, section 1.13.

If $\{u_n\}$ has finite number of distinct points and the rest are all equal, then let $u_n = u_m$ for $n \geqslant m$,

i.e. $u = \{u_1, u_2, u_3, \ldots, u_m, u_m, u_m, \ldots\}$.

Clearly every ϵ-neighbourhood of u_m contains all elements of u except perhaps $u_1, u_2, \ldots, u_{m-1}$ for

$$u_m - \epsilon < x < u_m + \epsilon \text{ for } x = u_n \text{ when } n \geqslant m.$$

$\therefore$ u_m is a limit point of the sequence $\{u_n\}$.

Note. (i) The limit of a sequence and limit point of a sequence are not the same thing. This is clear from the following example—

Let $\{u_n\}$ be a sequence such that

$$\left.\begin{aligned} & u_1 = 1,\ u_3 = 1,\ u_5 = 1,\ldots \\ \text{and}\quad & u_2 = 2,\ u_4 = 4,\ u_6 = 6,\ldots \end{aligned}\right\}$$

i.e.
$$u_{2n-1} = 1 \text{ for } n = 1, 2, 3, \ldots$$
$$u_{2n} = 2n \text{ for } n = 1, 2, 3 \ldots.$$

Here 1 is a limit point of the sequence but 1 is not a limit of the sequence. But if a sequence $\{u_n\}$ has only a finite number of equal terms then the limit is equal to the value of the limit point.

(ii) Converse of the above theorem is not true, for a sequence having a limit point need not be bounded. For example—

The sequence $\{u_n\}$ of the above example in note (i) has a limit point 1 but it is not bounded above.

From the above theorem we know a bounded sequence has at least one limit point. A bounded sequence may have more than one limit point. So for every bounded sequence we have a set of limit points (called derived set). As such the set limit points may or may not have a greatest or least ar as them.

Limit Superior and Limit Inferior.

Let $\{u_n\}$ be a sequence and S be the set of limit p f the sequence. The greatest element among the elements is called the limit superior (or upper limit) of the sequence nd it is denoted by $\overline{\lim}_{n\to\infty}\, u_n$ or $\overline{\lim}\, u_n$ or $\sup\limits_n u_n$.

The smallest element among the elements of S is called the limit inferior (or lower limit) of the sequence and it is denoted by $\underline{\lim}_{n\to\infty}\, u_n$ or $\underline{\lim}\, u_n$ or $\inf\limits_n u_n$.

The existence of limit superior and limit inferior for a bounded sequence is ensured by the following theorem.

Theorem 2.2. *The set of the limit points of a bounded set has the greatest and the smallest member.*

The proof of it is beyond the scope of the book.

that except a finite number of terms of the sequence, all the terms are situated very close to l on the number line R.

Note. (i) Some earlier authors use the word "however small" for ϵ which is superfluous as arbitrariness of ϵ includes that fact.

(ii) In the definition, m need not be taken an integer

$$\text{“}n \geqslant 5\tfrac{1}{8} \Rightarrow |\, u_n - l \,| < \epsilon\text{”}$$

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Clearly if for a sequence $\{u_n\}$, $u_n = l$ for infinite values of n then l is a limit point of the sequence.

***Theorem 2.1.** (Bolzano-Weierstrass Theorem)

Every bounded sequence has a limit point.

Proof. Let $\{u_n\}$ be a bounded sequence and u be the set of terms of the sequence i.e. $u = \{u_1, u_2, u_3, \ldots, u_n, \ldots\}$.

If $\{u_n\}$ has infinitely many distinct points then the result immediately follows from Bolzano-Weierstrass theorem on limit point in chapter 1, section 1.13.

If $\{u_n\}$ has finite number of distinct points and the rest are all equal, then let $u_n = u_m$ for $n \geqslant m$,

i.e. $$u = \{u_1, u_2, u_3, \ldots, u_m, u_m, u_m, \ldots\}.$$

Clearly every ϵ-neighbourhood of u_m contains all elements of u except perhaps $u_1, u_2, \ldots, u_{m-1}$ for

$$u_m - \epsilon < x < u_m + \epsilon \text{ for } x = u_n \text{ when } n \geqslant m.$$

$\therefore$ u_m is a limit point of the sequence $\{u_n\}$.

Note. (i) The limit of a sequence and limit point of a sequence are not the same thing. This is clear from the following example—

Let $\{u_n\}$ be a sequence such that

$$\left.\begin{array}{l} u_1 = 1,\ u_3 = 1,\ u_5 = 1, \dots \\ u_2 = 2,\ u_4 = 4,\ u_6 = 6, \dots \end{array}\right\}$$

and

i.e. $u_{2n-1} = 1$ for $n = 1, 2, 3, \dots$

$u_{2n} = 2n$ for $n = 1, 2, 3 \dots$

Here 1 is a limit point of the sequence but 1 is not a limit of the sequence. But if a sequence $\{u_n\}$ has only a finite number of equal terms then the limit is equal to the value of the limit point.

(ii) Converse of the above theorem is not true, for a sequence having a limit point need not be bounded. For example—

The sequence $\{u_n\}$ of the above example in note (i) has a limit point 1 but it is not bounded above.

From the above theorem we know a bounded sequence has at least one limit point. A bounded sequence may have more than one limit point. So for every bounded sequence we have a set of limit points (called derived set). As such the set of limit points may or may not have a greatest or least among them.

Limit Superior and Limit Inferior.

Let $\{u_n\}$ be a sequence and S be the set of limit points of the sequence. The greatest element among the elements of S is called the limit superior (or upper limit) of the sequence and it is denoted by $\overline{\lim_{n\to\infty}}\, u_n$ or $\overline{\lim}\, u_n$ or $\sup\limits_{n} u_n$.

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The existence of limit superior and limit inferior for a bounded sequence is ensured by the following theorem.

Theorem 2.2. *The set of the limit points of a bounded set has the greatest and the smallest member.*

The proof of it is beyond the scope of the book.

Example. (i) Let $\{u_n\}$ be a sequence for which $u_n = 1 + \dfrac{1}{n}$.

Clearly $1 < u_n < 2 \ \forall\, n \in N$. So $\{u_n\}$ is bounded and 1 is the only limit point. So the set S of limit points $= \{1\}$.

$\therefore$ Greatest element of S = least element of $S = 1$.

$\therefore$ Limit superior = limit inferior = 1.

(ii) Let $\{a_n\}$ be a sequence for which

$$a_n = (-1)^n\left(3 + \frac{1}{n}\right).$$

Clearly $-4 < a_n < 3{\cdot}5 \ \forall\, n \in N$. So $\{a_n\}$ is bounded. Limit points are -3 and 3. So the set of limit points

$$= \{-3, 3\}.$$

$\therefore$ Limit superior = 3, limit inferior = -3.

CONVERGENCE OF SEQUENCES

2.9. Convergent Sequence

(B U 1972, '77, '78; P U '76)

A sequence $\{u_n\}$ is said to be convergent to the limit l iff

$$u_n \to l \text{ as } n \to \infty.$$

Analytically, the sequence $\{u_n\}$ is said to be convergent to the limit l iff given arbitrary number $\epsilon > 0$ there exists a positive integer m such that

$$| u_n - l | < \epsilon \qquad \text{whenever } n \geqslant m$$

i.e. $$l - \epsilon < u_n < l + \epsilon \quad \text{whenever } n \geqslant m.$$

Example. Let us consider the sequence

$$\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots, \frac{2n-1}{2n}, \ldots$$

Now $$| u_n - 1 | = \left| \frac{2n-1}{2n} - 1 \right| = \left| -\frac{1}{2n} \right| = \frac{1}{2n}.$$

If ϵ be an arbitrarily small positive number, then

$$| u_n - 1 | < \epsilon, \text{ when } n > \frac{1}{2\epsilon}.$$

(Here $m = 1/2\epsilon$ or more strictly m is an integer which is just greater than $1/2\epsilon$.)

Hence the sequence $\{u_n\}$ is convergent and converges to the limit 1.

As already explained in case of a limit, the physical implication of saying, the sequence $\{u_n\}$ is convergent to l is to mean that all the terms of the sequence gather round l in very close vicinity except perhaps a finite number of them.

2.10. Divergent Sequence (B U 1972, '77, '78; P U '76)

If, corresponding to any positive number $\triangle$ that we may choose, no matter how great, there is a positive integer N, such that $u_n > \triangle$, for all $n \geqslant N$, then u_n is said to tend to infinity as n tends to infinity, and we write

$$u_n \to \infty \quad \text{or} \quad \lim u_n = \infty.$$

If, corresponding to an arbitrarily large negative number $-\triangle$, we can find out a positive integer N such that for every $n \geqslant N$, $u_n < -\triangle$, then u_n is said to tend to $-\infty$ as $n \to \infty$.

If $u_n \to \infty$ or $-\infty$, then the sequence $\{u_n\}$ is said to be divergent.

Examples. (i) Let us consider the sequence

$$1, 2, 3, 4, \ldots, n, \ldots$$

Let $\triangle$ be an arbitrarily large positive number, then

$$u_n > \triangle, \text{ when } n \geqslant \triangle + 1.$$

Thus $u_n \to +\infty$ as $n \to \infty$.

Hence the sequence $\{u_n\}$ is divergent and diverges to the limit $+\infty$.

(ii) Let us consider the sequence $\{u_n\}$,

i.e., $-1, -\sqrt{2}, -\sqrt{3}, \ldots, -\sqrt{n}, \ldots$

If $-\triangle$ be an arbitrarily large negative number,

$$u_n < -\triangle \text{ when } n \geqslant \triangle^2 + 1.$$

Hence $u_n \to -\infty$ as $n \to \infty$.

Therefore the sequence $\{u_n\}$ is divergent and diverges to the limit $-\infty$.

2.11. Oscillating Sequence

If u_n neither tends to a finite limit, nor to $+\infty$, nor to $-\infty$, it is said to oscillate and the sequence $\{u_n\}$ is said oscillatory.

If u_n oscillates and a positive number $\triangle$ can be found such that, for every positive integer n, $|u_n| < \triangle$ then u_n is said to oscillate finitely.

Otherwise, u_n oscillates infinitely.

Examples. (i) Let us consider the sequence $\{u_n\}$,

i.e., $-2, \frac{3}{2}, -\frac{4}{3}, \frac{5}{4}, -\frac{6}{5}, \ldots, (-1)^n\left(1+\frac{1}{n}\right) \ldots.$

When n is even,

$$|u_n - 1| = \left|1 + \frac{1}{n} - 1\right| = \left|\frac{1}{n}\right| = \frac{1}{n}.$$

If ϵ be an arbitrarily small positive number, then

$$|u_n - 1| < \epsilon, \text{ when } n > \frac{1}{\epsilon}.$$

Therefore $u_n \to 1$ as $n \to \infty$.

Again, when n is odd,

$$|u_n + 1| = \left|-1 - \frac{1}{n} + 1\right| = \left|-\frac{1}{n}\right| = \frac{1}{n}.$$

For an arbitrarily small positive number ϵ,

$$|u_n - (-1)| < \epsilon, \text{ when } n > \frac{1}{\epsilon}.$$

Therefore $u_n \to -1$ as $n \to \infty$.

Thus we find that the sequence $\{u_n\}$ does not tend to a definite limit as $n \to \infty$. Also $|u_n| < 3$ for all values of n.

Hence $\{u_n\}$ oscillates finitely.

(ii) Let us consider the sequence $\{u_n\} = \{(-1)^n n\}$,

i.e., $-1, 2, -3, 4, \ldots, (-1)^n n, \ldots$

Let n be even.

If $\triangle$ be an arbitrarily large positive number,

$$u_n > \triangle, \text{ when } n > \triangle.$$

Thus $u_n \to \infty$ as $n \to \infty$.

Let n be odd.

Then $\quad u_n \to \infty$ as $n \to \infty$.

Again $|u_n|$ is not less than a fixed positive number for all values of n.

Hence the sequence $\{u_n\}$ oscillates infinitely.

2.12. Cauchy's General Principle of Convergence of a Sequence

(P U 1976; Bh U '76; M U '85: B. U. '85; Mith. U. '85)

Theorem 2.3. *The necessary and sufficient condition for the convergence of the sequence* $\{a_n\}$ *is that corresponding to an arbitrary positive number* ϵ, *however small, there exists a positive integer* ν *such that* $|a_{n+p} - a_n| < \epsilon$, *for all values of* $n \geqslant \nu$ *and for all positive integral values of* p.

Proof. (i) Necessary Condition.

Let the sequence $\{a_n\}$ be convergent.

Then it must tend to a finite limit, say l.

Hence given a positive number ϵ, however small, there exists a positive integer ν such that

$$|a_n - l| < \frac{\epsilon}{2} \quad \cdot\cdot \quad (1),$$

for all $n \geqslant \nu$.

Then it follows that

$$|a_{n+p} - l| < \frac{\epsilon}{2} \quad \cdot\cdot \quad (2),$$

when $n \geqslant \nu$ and p is any positive integer.

$$\text{Now} \quad a_{n+p} - a_n = a_{n+p} - l - a_n + l$$
$$= (a_{n+p} - l) - (a_n - l).$$

$$\therefore \quad |a_{n+p} - a_n| = |(a_{n+p} - l) - (a_n - l)|$$
$$\leqslant |a_{n+p} - l| + |a_n - l| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

when $n \geqslant \nu$ and p is any + ve integer.

Hence $|a_{n+p} - a_n| < \epsilon$, when $n \geqslant \nu$ and p is any positive integer.

(ii) Sufficient Condition.

Let $| a_{n+p} - a_n | < \epsilon$, when $n \geqslant \nu$ and p is any positive integer.

i.e., $a_n - \epsilon < a_{n+p} < a_n + \epsilon$, when $n \geqslant \nu$ and p is any positive integer.

Thus the sequence $\{a_{n+p}\}$; $p = 1, 2, 3, \ldots$. is bounded.

If the greatest lower and the least upper bounds of it be m and M respectively, then $m > a_n - \epsilon$ and $M < a_n + \epsilon$.

$$\therefore \quad M - m < (a_n + \epsilon) - (a_n - \epsilon) = 2\epsilon.$$

Since ϵ is an arbitrarily positive number, however small,

we get $M - m = 0$, i.e., $M = m$.

$$\therefore \quad M - \epsilon < a_{n+p} < M + \epsilon,$$

i.e., $| a_{n+p} - M | < \epsilon$,

i.e., $a_{n+p} \to M$ as $p \to \infty$.

Hence the sequence $\{a_{n+p}\}$ and consequently the sequence $\{a_n\}$ converges to M.

2.13. Cauchy Sequence

A sequence $\{a_n\}$ is called a Cauchy sequence (or fundamental sequence) if given $\epsilon > 0$ there exists a positive integer ν such that $n, m \geqslant \nu \Rightarrow | a_n - a_m | < \epsilon$.

Theorem 2.4. *Every Cauchy sequence in R is a convergent sequence.*

Proof. Let $\{a_n\}$ be a Cauchy sequence of real numbers. So given $\epsilon > 0$ there exists $\nu \in N$. $n, m \geqslant \nu \Rightarrow | a_n - a_m | < \epsilon$.

Taking $m = n + p$ we get

$$n \geqslant \nu \Rightarrow | a_n - a_{n+p} | < \epsilon \quad \text{i.e.} \quad | a_{n+p} - a_n | < \epsilon.$$

Hence by Cauchy's general principle of convergence, $\{a_n\}$ is convergent.

Theorem 2.5. *Every convergent sequence in R is a Cauchy sequence.*

Proof. Let $\{a_n\}$ be a real sequence convergent to $l \in R$.

Then given $\epsilon > 0$ there exists $\mu \in N$ such that

$$n \geqslant \mu \Rightarrow | a_n - l | < \epsilon/2.$$

So, if $m \geqslant \mu \Rightarrow | a_m - l | < \frac{\epsilon}{2}$.

$$\therefore \quad | a_n - a_m | = | (a_n - l) - (a_m - l) |$$
$$\leqslant | a_n - l | + | a_m - l | < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$\therefore$ $\epsilon > 0$ there exists $n, m \geqslant \mu \Rightarrow | a_n - a_m | < \epsilon$.

Hence, by definition, $\{a_n\}$ is a Cauchy sequence.

Theorem 2.6. *Every Cauchy sequence in R is bounded.*

Proof. Let $\{a_n\}$ be a Cauchy sequence.

We have to prove that it is bounded.

By definition of Cauchy sequence, given $\epsilon > 0$ there exists $v \in N$. $n, m \geqslant v \Rightarrow | a_n - a_m | < \epsilon$.

Giving a particular value k for m greater than v,

$$| a_n - a_k | < \epsilon \text{ for } n \geqslant v.$$

$\therefore \quad a_k - \epsilon < a_n < a_k + \epsilon$ for $n \geqslant v$.

Let $\lambda = \text{max.} \{a_1, a_2, a_3, \ldots, a_{v-1}, a_k + \epsilon\}$

$\mu = \text{min.} \{a_1, a_2, a_3, \ldots, a_{v-1}, a_k - \epsilon\}$

Then $\mu < a_n < \lambda$ for all $n \in N$.

So $\{a_n\}$ is bounded. Thus every Cauchy sequence in R is bounded.

2.14. Some Properties of Convergent Sequences

Theorem 2.7. *A convergent sequence determines its limit uniquely.*

Proof. If possible, let the convergent sequence $\{ a_n \}$ has two distinct limits l and l'.

Since $\{a_n\}$ is convergent, therefore, by definition, there exists a positive integer m depending upon ϵ, an arbitrary positive number, however small, such that

$| a_n - l | < \epsilon/2$, for all values of $n \geqslant m$

and $| a_n - l' | < \epsilon/2$, for all values of $n \geqslant m$.

Now $l - l' = a_n - l' - a_n + l = (a_n - l') - (a_n - l)$.

$$\therefore \quad |l - l'| = |(a_n - l') - (a_n - l)|$$
$$\leqslant |a_n - l'| + |a_n - l| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

i.e., $|l - l'| < \epsilon$, for all values of $n \geqslant m$.

But ϵ is an arbitrary positive number, however small. So ultimately $l = l'$.

Hence the limit is unique.

Theorem 2.8. *Every convergent sequence is bounded.*
(P U 1968, '69 H; R U '66 H; M U '68 H, '80 H; Mith U '82, '85)

Proof. Let $\{a_n\}$ be a convergent sequence. So a_n must tend to a definite limit l as $n \to \infty$.

Then, by definition, there exists a positive integer m depending upon an arbitrary positive number ϵ, however small, such that

$$|a_n - l| < \epsilon, \text{ for all values of } n \geqslant m;$$

i.e., $$l - \epsilon < a_n < l + \epsilon, \text{ for all } n \geqslant m. \quad .. \quad (1)$$

Now take the elements $a_1, a_2, a_3, \ldots a_{n-1}$ which are finite in number.

Suppose G and g are the greatest and the least elements among them.

$$\left.\begin{array}{l}\text{Let } M = \text{greatest of } G \text{ and } l + \epsilon \\ \text{and } m = \text{least of } g \text{ and } l - \epsilon\end{array}\right\}. \quad .. \quad (2)$$

From (1) and (2), we obtain

$$m \leqslant a_n \leqslant M \text{ for all values of } n.$$

Hence, by the definition of bounds, the sequence $\{a_n\}$ is bounded.

[**Note.** The converse need not be true i.e., there are bounded sequences which are not convergent. For example—

If $u_n = (-1)^n$, the sequence $\{u_n\}$ is bounded for

$$u_n = -1 \text{ or } 1 \ \forall\ n \in N.$$

But $\{u_n\}$ has no limit and so it is not convergent.]

2.15. Convergence of Monotone Sequences

Theorem 2.9. *Every monotonic increasing sequence tends to its least upper bound.* (Bh U 1969 H; M U '63 H, '67 H, '75 H; B U '55 H, '59 H, '61 H, '70, '75; P U '66, '70)

Proof. Let $\{a_n\}$ be a monotonic increasing sequence whose least upper bound is M.

Then, by the definition of upper bound,

$$a_n \leqslant M, \text{ for all values of } n, \quad \ldots \quad (1)$$

and $a_n + \epsilon > M$, for at least one value of n.

Let this be true for $n = m$.

Then $a_m + \epsilon > M$; i.e., $a_m > M - \epsilon$. .. (2)

Since the sequence is monotonic increasing, therefore

$$a_n \geqslant a_m, \text{ when } n \geqslant m. \quad \ldots \quad (3)$$

From (2) and (3), $a_n > M - \epsilon$, for all $n \geqslant m$.

Again from (1), $a_n \leqslant M$, for all n.

So, $a_n < M + \epsilon$, for all n and hence also for $n \geqslant m$.

Combining these results, we obtain

$$M - \epsilon < a_n < M + \epsilon, \text{ for } n \geqslant m.$$

$$\therefore \; | a_n - M | < \epsilon, \text{ for } n \geqslant m.$$

Hence $\{ a_n \}$ tends to M.

Hence the theorem.

Theorem 2.10. *Every monotonic increasing sequence, which is not bounded above, diverges to* $+\infty$.

Proof Let $\{a_n\}$ be a monotonic increasing sequence which is not bounded above.

This means that, given any number $\triangle$

$$a_n > \triangle,$$

for at least one value of n, say $n = m$, so that $a_m > \triangle$.

Since $\{a_n\}$ is monotonic increasing, we must have, for all value of $n \geqslant m$, $a_n \geqslant a_m$.

$$\therefore \; a_n > \triangle \text{ for all values of } n \geqslant m.$$

In other words $a_n \to \infty$ as $n \to \infty$.

Theorem 2.11. *Every monotonic decreasings equence tends to its greatest lower bound.* (P U 1966; Bh U '72 H)

Proof. Let $\{a_n\}$ be a monotonic decreasing sequence whose greatest lower bound is K.

Then, by the definition of lower bound,

$$a_n \geqslant K, \text{ for all values of } n, \quad \ldots \quad (1)$$

and $a_n < K + \epsilon$, for at least one value of n.

Let it be true for $n = m$.

Then $$a_m < K + \epsilon. \quad .. \quad (2)$$

Since $\{a_n\}$ is a monotonic decreasing sequence, therefore

$$a_n \leqslant a_m, \text{ for } n \geqslant m. \quad .. \quad (3)$$

From (2) and (3), $a_n < K + \epsilon$, for $n \geqslant m$.

Again, from (1), $a_n > K - \epsilon$ for all n and so for $n \geqslant m$.

Combining these results, we obtain

$$K - \epsilon < a_n < K + \epsilon, \text{ for } n \geqslant m;$$

i.e., $$| a_n - K | < \epsilon, \text{ for } n \geqslant m.$$

Hence $\{a_n\}$ tends to the greatest lower bound K.

Theorem 2.12. *Every monotonic decreasing sequence not bounded below, diverges to* $-\infty$.

Proof. Let $\{a_n\}$ be a monotonic decreasing sequence not bounded below. So given any negative number $-\triangle$,

$$a_n < -\triangle \text{ for at least one value of } n.$$

Let $a_m < -\triangle$.

As $\{a_n\}$ is m.d. we get $a_n \leqslant a_m$ for all $n \geqslant m$.

$$\therefore \quad a_n \leqslant a_m < -\triangle \text{ for all } n \geqslant m.$$

This means $a_n \to -\infty$ as $n \to \infty$.

2.16. Necessary and Sufficient Condition for Monotone Sequences to be Convergent

Theorem 2.13. *The necessary and sufficient condition for a monotone sequence to be convergent is that it is bounded.*

Proof. Let $\{u_n\}$ be a monotonic sequence which is convergent.

As in theorem 2.8, we can prove $\{u_n\}$ is bounded.

$\therefore$ Boundedness is a necessary condition.

Next let $\{u_n\}$ be a bounded monotonic sequence.

For definiteness let $\{u_n\}$ be monotonic increasing whose least upper bound is M.

As in theorem 2.9, we can prove that $\{u_n\}$ converges to M.

If $\{u_n\}$ be monotonic decreasing whose greatest lower bound is m, then by theorem 2.11 we can prove that $\{u_n\}$ converges to m.

$\therefore$ The necessary and sufficient condition for a monotonic sequence to be convergent is that it is bounded.

2.17. Combination Theorems on Limit

Theorem 2.14. *If* $\lim_{n \to \infty} u_n = l$ *and* $\lim_{n \to \infty} v_n = l'$, *then*

$$\lim_{n \to \infty} (u_n + v_n) = l + l'. \qquad \text{(M U 1968 H)}$$

Proof. Since $u_n \to l$ and $v_n \to l'$ as $n \to \infty$, therefore given $\epsilon > 0$, however small, there exist two positive integers m_1 and m_2 such that

$$| u_n - l | < \frac{\epsilon}{2}, \text{ for all } n \geqslant m_1 \quad .. \quad (1)$$

and
$$| v_n - l' | < \frac{\epsilon}{2}, \text{ for all } n \geqslant m_2. \quad .. \quad (2)$$

Let m be a positive integer greater than m_1 and m_2. Then the inequalities (1) and (2) both hold when $n \geqslant m$.

Since
$$\begin{aligned} | (u_n + v_n) - (l + l') | &= | (u_n - l) + (v_n - l') | \\ &\leqslant | u_n - l | + | v_n - l' | \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ when } n \geqslant m. \end{aligned}$$

Hence $(u_n + v_n) \to l + l'$ as $n \to \infty$.

Note. *Similarly we can prove that* $(u_n - v_n) \to l - l'$ *when* $u_n \to l$ *and* $v_n \to l'$.

Theorem 2.15. *If* $\lim_{n \to \infty} u_n = l$ *and* $\lim_{n \to \infty} v_n = l'$, *then*

$$\lim_{n \to \infty} (u_n v_n) = ll'. \qquad \text{(R U 1970 H)}$$

Proof. As $u_n \to l$, $v_n \to l'$ we get $\{u_n\}$ and $\{v_n\}$ are convergent and bounded sequence.

$$\therefore \quad | u_n | \leqslant k, \quad | v_n | \leqslant k'.$$

Now
$$\begin{aligned} | u_n v_n - ll' | &= | u_n(v_n - l') + l'(u_n - l) | \\ &\leqslant | u_n(v_n - l') | + | l'(u_n - l) | \end{aligned}$$

$$\therefore \quad | u_n v_n - ll' | \leqslant | u_n | \, | v_n - l' | + | l' | \, | u_n - l |$$
$$\leqslant k \, | v_n - l' | + | l' | \, | u_n - l | . \quad .. \quad (1)$$

As $u_n \to l$, $u_n \to l'$ by definition we get, given $\epsilon > 0$,

$$| u_n - l | < \frac{\epsilon/2}{1 + | l' |} \text{ for all } n \geqslant m$$

$$| v_n - l' | < \frac{\epsilon/2}{k} \text{ for all } n \geqslant m'.$$

Both these hold for $n \geqslant \nu = \max \{m, m'\}$.

$\therefore$ From (1),

$$| u_n v_n - ll' | < \frac{\epsilon}{2} + | l' | \; \frac{\epsilon/2}{1 + | l' |} \text{ for all } n \geqslant \nu$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

$\therefore$ $u_n v_n \to ll'$ i.e. $\lim\limits_{n \to \infty} (u_n v_u) = ll'$.

Cor. If $u_n \to l$, then $(ku_n) \to kl$, where k is a constant.

Theorem 2.16. *If* $\lim\limits_{n \to \infty} u_n = l$, $\lim\limits_{n \to \infty} \dfrac{1}{u_n} = \dfrac{1}{l}$, *provided* $l \neq 0$.

Proof. Let $u_n = l + \alpha$, where $\alpha \to 0$ as $n \to \infty$.

Now, $\dfrac{1}{u_n} - \dfrac{1}{l} = \dfrac{1}{l + \alpha} - \dfrac{1}{l} = - \dfrac{\alpha}{l(l + \alpha)}$.

$$\therefore \left| \frac{1}{u_n} - \frac{1}{l} \right| = \left| - \frac{\alpha}{l(l + \alpha)} \right|$$
$$= \frac{| \alpha |}{| l | \; | l + \alpha |} < \frac{| \alpha |}{| l | L},$$

where $0 < L < | l + \alpha |$.

For an arbitrarily small positive number ϵ, we may choose a positive integer m such that

$$| \alpha | < | l | L . \epsilon \text{ when } n \geqslant m \quad \{\because \alpha \to 0 \text{ as } n \to \infty\}$$

Thus $\left| \dfrac{1}{u_n} - \dfrac{1}{l} \right| < \epsilon$, when $n \geqslant m$.

Hence $\frac{1}{u_n} \to \frac{1}{l}$ as $n \to \infty$.

Theorem 2.17. *If* $\lim\limits_{n\to\infty} u_n = l$ *and* $\lim\limits_{n\to\infty} v_n = l'$, *then*

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{l}{l'}, \text{ provided } l' \neq 0.$$

(M U 1971 H)

Proof. From the last two theorems, we can easily prove that

$$\frac{u_n}{v_n} = u_n \cdot \frac{1}{v_n} \to l.\frac{1}{l'}, \text{ when } l' \neq 0.$$

Theorem 2.18. (Cauchy's First Theorem on Limits)

(M U 1980 H)

If $\lim a_n = l$, *when* $n \to \infty$, *then*

$$\lim_{n\to\infty} \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} = l.$$

Proof. Let $a_n = l + \epsilon_n$, so that $\epsilon_n \to 0$ as $a_n \to l$.

Since the sequence $\{a_n\} \to l$, therefore the sequence $\{\epsilon_n\} \to 0$.

Now, $\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}$

$$= \frac{(l+\epsilon_1) + (l+\epsilon_2) + (l+\epsilon_3) + \ldots + (l+\epsilon_n)}{n}$$

$$= \frac{nl}{n} + \frac{\epsilon_1 + \epsilon_2 + \epsilon_3 + \ldots + \epsilon_n}{n}$$

$$= l + \frac{\epsilon_1 + \epsilon_2 + \epsilon_3 + \ldots + \epsilon_n}{n}.$$

Thus we have to prove that

$$\frac{\epsilon_1 + \epsilon_2 + \epsilon_3 + \ldots + \epsilon_n}{n} \to 0 \text{ as } \epsilon_n \to 0.$$

Since $\{\epsilon_n\}$ is convergent, it is bounded and there exists a fixed positive number k such that

$$|\epsilon_n| < k, \text{ for all } n. \quad \ldots \quad (1).$$

Since $\{\epsilon_n\}$ converges to zero, therefore for a given positive number ϵ, however small, there exists a positive integer m such that $|\epsilon_n| < \frac{\epsilon}{2}$ for all $n \geqslant m$. .. (2)

Thus we have $\left| \frac{\epsilon_1 + \epsilon_2 + \epsilon_3 + \ldots + \epsilon_n}{n} \right|$

$$= \left| \frac{\epsilon_1 + \epsilon_2 + \epsilon_3 + \ldots + \epsilon_m + \epsilon_{m+1} + \epsilon_{m+2} + \ldots + \epsilon_n}{n} \right|$$

$$\leqslant \left| \frac{\epsilon_1 + \epsilon_2 + \epsilon_3 + \ldots + \epsilon_m}{n} \right| + \left| \frac{\epsilon_{m+1} + \epsilon_{m+2} + \ldots + \epsilon_n}{n} \right|$$

$$< \frac{mk}{n} + \frac{n-m}{n} \cdot \frac{\epsilon}{2}, \text{ by virtue of (1) and (2)}$$

$$< \frac{mk}{n} + \frac{\epsilon}{2}.$$

Keeping m fixed, we find that

$$\frac{mk}{n} < \frac{\epsilon}{2} \text{ if } n > \frac{2mk}{\epsilon}.$$

Let μ be any integer $> \frac{2mk}{\epsilon}$ so that for $n \geqslant \mu$ we have

$$\frac{mk}{n} < \frac{\epsilon}{2}.$$

Let λ be a positive integer greater than both m and μ; then

$$\left| \frac{\epsilon_1 + \epsilon_2 + \epsilon_3 + \ldots + \epsilon_n}{n} \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

for every $n \geqslant \mu$.

$$\therefore \lim \frac{\epsilon_1 + \epsilon_2 + \epsilon_3 + \ldots + \epsilon_n}{n} = 0, \text{ as } n \to \infty.$$

Hence $\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} \to l$, as $n \to \infty$.

Note. The converse of this theorem is not necessarily true.

Let $a_n = (-1)^n$. Then the sequence

$$\{a_n\} \equiv -1,\ 1,\ -1,\ 1, \ldots$$

which is not convergent since a_n is -1 or 0 according as n is odd or even and consequetly has no limit. But

$$\frac{a_1 + a_2 + \ldots + a_n}{n} = \frac{-1 + 1 - 1 + \ldots + (-1)^n}{n}$$

$$= -\frac{1}{n} \text{ or } 0$$

according as n is odd or even and hence this → 0.

Note. Cauchy's Second Theorem on Limits.

Statement. If $\{a_n\}$ be a sequence of positive terms, then

$$\lim_{n \to \infty} (a_n)^{\frac{1}{n}} = \lim_{n \to \infty} \left(\frac{a_{n+1}}{a_n}\right),$$

provided $\lim\limits_{n \to \infty} \dfrac{a_{n+1}}{a_n}$ exists, whether finite or infinite.

2.18*. Subsequence

Let $\{a_n\}$ be a sequence. Then the sequence $\{a_{n_r}\}$, where $\{n_r\}$ is a strictly monotonic sequence of natural numbers, is called a subsequence of the sequence $\{a_n\}$.

Example. (i) $\{a_{2n}\}$ is a subsequence of the sequence $\{a_n\}$ because $\{2n\}$ is strictly monotonic sequence of natural numbers.

(ii) The sequence $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$ is a subsequence of the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

Theorem 2.19. *Every subsequence of a convergent sequence is convergent to the same limit.*

Proof. Let the sequence $\{a_n\}$ converges to the limit l. By definition,

$\epsilon > 0$ there exists $m \in N$. $n \geqslant m \Rightarrow |a_n - l| < \epsilon$. .. (1)

Let $\{a_{n_r}\}$ be a subsequence of the sequence $\{a_n\}$.

We have to prove that $a_{n_r} \to l$ as $r \to \infty$.

Now if $r > m$, $n_r \geqslant r > m$.

So by (1), $| a_{n_r} - l | < \epsilon$.

Thus $\epsilon > 0$ there exists $m \in N \;.\; n_r \geqslant m \Rightarrow | a_{n_r} - l | < \epsilon$.

$\therefore\; a_{n_r} \to l$ as $r \to \infty$.

$\therefore\; \{a_{n_r}\}$ is convergent to the same limit l.

Theorem 2.20. *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. Let $\{a_n\}$ be a bounded sequence of real numbers.

Now we construct a subsequence $\{a_{n_r}\}$ of the sequence $\{a_n\}$ in the following manner—

If every set $\{a_n \mid n > m,\; m \in N\}$ has a maximum, take

$$a_{n_1} = \max_{n > 1} a_n$$

$$a_{n_2} = \max_{n > n_1} a_n$$

$$a_{n_3} = \max_{n > n_2} a_n \quad \text{and so on.}$$

Now consider the sequence $\{a_{n_r}\}$. Clearly

$$n_1 < n_2 < n_3 < \ldots.$$

Also $a_{n_1} > a_{n_2} > a_{n_3} \ldots$ by construction.

Thus $\{a_{n_r}\}$ is a monotonic decreasing subsequence of the sequence $\{a_n\}$.

If at least one of the sets $\{a_n \mid n > m,\; m \in N\}$, does not have a maximum, then let for some m_1, $\{a_n \mid n > m_1\}$ does not have a maximum.

$\therefore$ If $\lambda > m_1$ we can find an a_n coming after a_λ such that $a_n > a_\lambda$ for otherwise the greatest among $a_{m_{1+1}}, a_{m_{1+2}}, \ldots, a_\lambda$ would be the maximum element of $\{a_n \mid n > \lambda\}$.

Let $a_{n_1} = a_{m_1+1}$

$a_{n_2} =$ the first term after a_{n_1} for which $a_{n_2} > a_{n_1}$

$a_{n_3} =$ the first term after a_{n_2} for which $a_{n_3} > a_{n_2}$

and so on.

Clearly $n_1 < n_2 < n_3 < \ldots$ and $a_{n_1} < a_{n_2} < a_{n_3} < \ldots$

So $\{a_{n_r}\}$ is a monotonic increasing subsequence of $\{a_n\}$.

Thus in all cases we can have a monotonic subsequence $\{a_{n_r}\}$ of the bounded sequence $\{a_n\}$.

Hence $\{a_{n_r}\}$ will be a bounded monotonic sequence and consequently will be convergent (by theorem 2.13).

WORKED OUT EXAMPLES

1. *If* $u_n = \dfrac{2n+1}{3n+5}$, *show that* $\lim\limits_{n\to\infty} u_n = \frac{2}{3}$, *by definition of limit.*

(यदि $u_n = \dfrac{2n+1}{3n+5}$, तो सीमा की परिभाषा से सिद्ध करें कि

$$\lim_{n\to\infty} u_n = \tfrac{2}{3}.)$$

Solution. By definition, $\frac{2}{3}$ will be the limit of u_n provided for a given $\epsilon > 0$, we are able to find a positive integer m such that for all $n \geqslant m$, we have $|u_n - \frac{2}{3}| < \epsilon$

i.e., if $$\left|\frac{2n+1}{3n+5} - \frac{2}{3}\right| < \epsilon$$

i.e., if $$\left|\frac{-7}{9n+15}\right| < \epsilon$$

i.e., if $$\frac{7}{9n+15} < \epsilon$$

i.e., if $$9n + 15 > \frac{7}{\epsilon}$$

i.e., if $$n > \frac{7}{9\epsilon} - \frac{5}{3}.$$

Taking m equal to a positive integer greater than $\dfrac{7}{9\epsilon} - \dfrac{5}{3}$, we see that $|u_n - \frac{2}{3}| < \epsilon$ is satisfied for all $n \geqslant m$.

Hence $$\lim_{n\to\infty} u_n = \tfrac{2}{3}.$$

2. *Show, with the help of Cauchy's general principle of convergence, that the sequence f where*

$$f(n) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

is not convergent.

(कोशि के 'अभिसरण के सामान्य सिद्धान्त' से सिद्ध करें कि अनुक्रम f अभिसारी नहीं है जहाँ $f(n) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$.)

Solution. Let us suppose that the sequence converges; then there exists a positive integer m such that for every $n \geqslant m$ and for every $p \geqslant 0$, we have $| f(n+p) - f(n) | < \epsilon$, where ϵ is an arbitrarily small quantity,

i.e., $| f(n+p) - f(n) |$

$$= \left| \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+p} \right| < \epsilon. \quad .. \quad (1)$$

In particular, when $n = m$ and $p = m$, we find that

$$\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+p}$$

$$= \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{m+m}$$

$$> \frac{1}{m+m} + \frac{1}{m+m} + \dots + \frac{1}{m+m}$$

$$= \frac{m}{2m} = \frac{1}{2}.$$

Now ϵ is at out choice and can be taken as small as we please.

Let $\epsilon < \frac{1}{2}$.

Thus we have a contradiction. Hence the sequence cannot converge.

3. *Prove that the sequence $\{u_n\}$ converges to a limit lying between 2 and 3, where $u_n = \left(1 + \frac{1}{n}\right)^n$.*

(P U 1966, '68, '76, '85; B U '74)

[सिद्ध करें कि अनुक्रम $\{u_n\}$, 2 और 3 के बीच स्थित एक सीमा की ओर अभिसारी हैं जहाँ $u_n = \left(1 + \frac{1}{n}\right)^n$].

Proof. Here $u_n = \left(1 + \frac{1}{n}\right)^n$.

Using Binomial Theorem, we have

$$u_n = 1 + n.\frac{1}{n} + \frac{n(n-1)}{\underline{|2}} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{\underline{|3}} \cdot \frac{1}{n^3} + \ldots + \frac{n(n-1)(n-2)\ldots 1}{\underline{|n}} \cdot \frac{1}{n^n}.$$

$$= 1 + 1 + \frac{1}{\underline{|2}}\left(1 - \frac{1}{n}\right) + \frac{1}{\underline{|3}}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \frac{1}{\underline{|n}}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\ldots\left(1 - \frac{n-1}{n}\right).$$

Obviously, $u_n > 2$

and $u_n < 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \ldots + \frac{1}{\underline{|n}}$,

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^{n-1}},$$

i.e., $u_n < 1 + \dfrac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 1 + 2\left(1 - \frac{1}{2^n}\right) < 1 + 2.1 = 3.$

$\therefore$ $u_n < 3$, for all n.

As n increases, the number of terms in u_n increases.

Also each factor in each term (except the first two terms) increases as n increases, for $\frac{1}{n}, \frac{2}{n}, \ldots$ decrease as n increases.

$\therefore$ u_n goes on increasing as n increases.

So $\{u_n\}$ is a monotonic increasing sequence bounded above by 3 and is therefore convergent.

Note. Since $2 < u_n < 3$, for all n, therefore

$$2 < \lim_{n \to \infty} u_n < 3,$$

showing that the limit of u_n lies between 2 and 3.

Thus if $\lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$, then $2 < e < 3$.

4. *Prove that the sequence $\{a_n\}$ defined by the relation*

$$a_n = 1 + \frac{1}{\lfloor\underline{1}} + \frac{1}{\lfloor\underline{2}} + \frac{1}{\lfloor\underline{3}} + \ldots + \frac{1}{\lfloor\underline{n-1}}$$

convergent.

(सिद्ध करें कि सम्बन्ध

$$a_n = 1 + \frac{1}{\lfloor\underline{1}} + \frac{1}{\lfloor\underline{2}} + \frac{1}{\lfloor\underline{3}} + \ldots + \frac{1}{\lfloor\underline{n-1}}$$

द्वारा परिभाषित अनुक्रम $\{a_n\}$ अभिसारी है।)

Proof. Here $a_n = 1 + \frac{1}{\lfloor\underline{1}} + \frac{1}{\lfloor\underline{2}} + \ldots + \frac{1}{\lfloor\underline{n-1}}$

and $\quad a_{n+1} = 1 + \frac{1}{\lfloor\underline{1}} + \frac{1}{\lfloor\underline{2}} + \ldots + \frac{1}{\lfloor\underline{n-1}} + \frac{1}{\lfloor\underline{n}}$.

$$\therefore \quad a_{n+1} - a_n = \frac{1}{\lfloor\underline{n}} > 0, \text{ for all } n.$$

Thus the sequence $\{a_n\}$ is monotonically increasing.

Clearly $\lfloor\underline{n} = (1.2.3.\ldots.n) \geqslant 2^{n-1}$, whenever $n \geqslant 1$

and $\quad \lfloor\underline{n-1} \geqslant 2^{n-2}$, whenever $n \geqslant 2$.

$$\therefore \quad a_n \leqslant 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-2}}$$

i.e., $\quad a_n \leqslant 1 + \dfrac{1 - \frac{1}{2^{n-1}}}{1 - \frac{1}{2}}$

$$= 1 + 2\left(1 - \frac{1}{2^{n-1}}\right) < 1 + 2.1 = 3.$$

$\therefore \quad a_n < 3$, whenever $n \geqslant 2$.

Also, $\quad a_1 = 1 + \frac{1}{\lfloor\underline{0}} = 1 + \frac{1}{1} = 1 + 1 = 2 < 3$.

$$\therefore \quad a_n < 3 \text{ for all } n.$$

Since $a_n > 0$, for all n, therefore, $0 < a_n < 3$, for all n.

Thus the sequence $\{a_n\}$ is bounded.

Since the sequence $\{a_n\}$ is bounded and monotonically increasing, therefore it must be convergent.

Note. Similarly we can prove that the sequence $\{a_n\}$, where

$$a_n = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \ldots + \frac{1}{\lfloor n},$$

is convergent.

5. *If* $a_n = \dfrac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}$, *show that the sequence* $\{a_n\}$ *is monotonic and hence deduce its limit.* (P U 1970)

(यदि $a_n = \dfrac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}$, तो सिद्ध करें कि अनुक्रम $\{a_n\}$ एकदिष्ट है और फिर इसकी सीमा निकालें।)

Solution. $a_{n+1} = \dfrac{1.3.5\ldots(2n-1)(2n+1)}{2.4.6\ldots 2n(2n+2)}$.

$$\therefore \quad \frac{a_{n+1}}{a_n} = \frac{1.3.5\ldots(2n-1)(2n+1)}{2.4.6\ldots 2n(2n+2)} \times \frac{2.4.6\ldots 2n}{1.3.5\ldots(2n-1)}$$

$$= \frac{2n+1}{2n+2} < 1; \quad \therefore \quad a_{n+1} < a_n \quad \forall\, n \in N.$$

$\therefore$ $\{a_n\}$ is a monotonic decreasing sequence.

As $\quad a_n > 0 \ \forall\, n \in N.$

$\therefore$ $\{a_n\}$ is bounded below. Hence $\{a_n\}$ is convergent.

Again $\quad \dfrac{1}{2} < \dfrac{2}{3}, \dfrac{3}{4} < \dfrac{4}{5}, \ldots, \dfrac{2n-1}{2n} < \dfrac{2n}{2n+1}$.

$$\therefore \quad a_n = \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n} < \frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n+1}.$$

$$\therefore \quad 0 < a_n^2 < \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n} \times \frac{2.4.6\ldots 2n}{3.5.7\ldots(2n+1)}$$

$$= \frac{1}{2n+1}.$$

But $\dfrac{1}{2n+1} \to 0$ as $n \to \infty$. Hence $\lim\limits_{n \to \infty} a_n = 0$.

6. *If S_1 and S_2 are positive and $S_{n+1} = \frac{1}{2}(S_n + S_{n-1})$, for $n \geqslant 2$, prove that the sequences $\{S_{2n+1}\}$ and $\{S_{2n}\}$ are both monotone, one increasing and the other decreasing. Further, prove that both the sequences converge to the same limit and find the limit.*

[यदि S_1 और S_2 धनात्मक हों एवं $S_{n+1} = \frac{1}{2}(S_n + S_{n-1})$ जबकि $n \geqslant 2$ तो सिद्ध करें कि अनुक्रम $\{S_{2n+1}\}$ और $\{S_{2n}\}$ दोनों एकदिष्ट हैं जिनमें एक वर्धमान एवं एक ह्रासमान है। फिर सिद्ध करें कि दोनों अनुक्रम एक ही सीमा की ओर अभिसारी हैं एवं उस सीमा को निकालें।]

(M U 1965 H, '75 H; Bh U '68 H; R U '67 H;
B U '54 H, '67 H; P U '62 H)

Solution. We have $S_{n+1} = \frac{1}{2}(S_n + S_{n-1})$.

Putting $n = 2$, we get $S_3 = \frac{1}{2}(S_2 + S_1)$.

Hence S_3 is the arithmetic mean between S_1 and S_2.

CASE I. Let $S_1 > S_2$.

Then
$$S_1 > S_3 > S_2. \qquad \text{.. } (1)$$

Again putting $n = 3$, we get $S_4 = \frac{1}{2}(S_3 + S_2)$.

Hence S_4 is the A.M. between S_2 and S_3.
$$S_3 > S_4 > S_2, \text{ as, from (1), } S_3 > S_2. \qquad \text{.. } (2)$$

From (1) and (2), we find that
$$S_1 > S_3 > S_4 > S_2. \qquad \text{.. } (3)$$

Putting $n = 4$, we get $S_5 = \frac{1}{2}(S_4 + S_3)$.

$\therefore$ S_5 is the A.M. between S_3 and S_4.

$\therefore$ $S_3 > S_5 > S_4$, as, from (2), $S_3 > S_4$. .. (4)

From (3) and (4), we get
$$S_1 > S_3 > S_5 > S_4 > S_2.$$

Repeating this process, we get
$$S_1 > S_3 > S_5 > S_7 > \ldots. > S_6 > S_4 > S_2.$$

CASE II. If $S_1 < S_2$, then $S_1 < S_3 < S_2$, just like Case I.

Also, $S_1 < S_3 < S_4 < S_2$, like Case I.

Again $S_1 < S_3 < S_5 < S_4 < S_2$, like Case I.

.. ..
.. ..
.. ..

$$S_1 < S_3 < S_5 < S_7 < \ldots. < S_6 < S_4 < S_2.$$

In both the cases the sequences $\{S_{2n+1}\}$ and $\{S_{2n}\}$ are monotone, one increasing and the other decreasing.

Both are bounded by S_1, S_2 and hence they must be convergent.

Let $\lim\limits_{n\to\infty} S_{2n+1} = l$

$$\Rightarrow \lim_{n\to\infty} S_{2n-1} = l, \text{ and let } \lim_{n\to\infty} S_{2n} = m.$$

$$\text{Now} \quad S_{2n+1} = \tfrac{1}{2}(S_{2n} + S_{2n-1})$$

$$\Rightarrow \lim_{n\to\infty} S_{2n+1} = \tfrac{1}{2}\left(\lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} S_{2n-1}\right)$$

$$\Rightarrow \quad l = \tfrac{1}{2}(m + l)$$

$$\Rightarrow \quad 2l = m + l \quad \Rightarrow \quad l = m.$$

Hence both the sequences tend to the same limit.

$$\begin{aligned} \because \quad 2S_{2n+1} &= S_{2n} + S_{2n-1}, \\ 2S_{2n} &= S_{2n-1} + S_{2n-2} \\ 2S_{2n-1} &= S_{2n-2} + S_{2n-3} \\ \cdot\cdot \quad \cdot\cdot \quad &\cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \\ \cdot\cdot \quad \cdot\cdot \quad &\cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \\ \cdot\cdot \quad \cdot\cdot \quad &\cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \end{aligned}$$

$$\begin{aligned} \therefore \quad 2S_{2n+1} + S_{2n} &= S_{2n} + S_{2n-1} + S_{2n} \\ &= 2S_{2n} + S_{2n-1} \\ &= S_{2n-1} + S_{2n-2} + S_{2n-1} \\ &= 2S_{2n-1} + S_{2n-2} \\ &= S_{2n-2} + S_{2n-3} + S_{2n-2} \\ &= 2S_{2n-2} + S_{2n-3} \\ &= \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \quad \cdot\cdot \\ &= 2S_2 + S_1. \end{aligned}$$

$$\therefore \quad 2\lim_{n\to\infty} S_{2n+1} + \lim_{n\to\infty} S_{2n} = 2S_2 + S_1$$

$$\Rightarrow \quad 2l + l = 2S_2 + S_1$$

$$\Rightarrow \quad 3l = 2S_2 + S_1.$$

$$\text{Hence} \quad l = \tfrac{1}{3}(2S_2 + S_1).$$

This is the required limit.

Alternative Method.

As $S_{n+1} = \frac{1}{2}(S_n + S_{n-1})$ we get $S_{n+1} - S_n = \frac{1}{2}(S_{n-1} - S_n)$ for $n \geqslant 2$.

$$\begin{aligned}\therefore \quad | S_{n+1} - S_n | &= \tfrac{1}{2} | S_n - S_{n-1} | = \tfrac{1}{2} | \tfrac{1}{2}(S_{n-2} - S_{n-1}) | \\ &= \frac{1}{2^2} | S_{n-1} - S_{n-2} | \\ &= \frac{1}{2^2} \left| \frac{1}{2}(S_{n-3} - S_{n-2}) \right| \\ &= \frac{1}{2^3} | S_{n-2} - S_{n-3} | \\ &= \ldots = \frac{1}{2^{n-1}} | S_2 - S_1 | \text{ for all } n \geqslant 2.\end{aligned}$$

Now if $n > m$ (i.e. $n = m + p$),

$$\begin{aligned}| S_n - S_m | & \\ &= | (S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + \ldots + (S_{m+1} - S_m) | \\ &\leqslant | S_n - S_{n-1} | + | S_{n-1} - S_{n-2} | + \ldots + | S_{m+1} - S_m |, \\ &\qquad\qquad \{\because \ | a + b | \leqslant | a | + | b | \} \\ &= \frac{1}{2^{n-2}} | S_2 - S_1 | + \frac{1}{2^{n-3}} | S_2 - S_1 | + \ldots \\ &\qquad\qquad + \frac{1}{2^{m-1}} | S_2 - S_1 | \\ &= \left(\frac{1}{2^{n-2}} + \frac{1}{2^{n-3}} + \ldots + \frac{1}{2^{m-1}} \right) | S_2 - S_1 | \\ &= \frac{\frac{1}{2^{n-2}}(1 - 2^{n-m})}{1 - 2} | S_2 - S_1 | \\ &= \frac{1}{2^{n-2}} (2^{n-m} - 1) | S_2 - S_1 | \\ &= \left(\frac{1}{2^{m-2}} - \frac{1}{2^{n-2}} \right) | S_2 - S_1 | \\ &\leqslant \frac{1}{2^{m-2}} | S_2 - S_1 | . \quad .. \qquad (1)\end{aligned}$$

Given $\epsilon > 0$ we select ν such that $\frac{1}{2^{\nu-2}} | S_2 - S_1 | < \epsilon$.

Then from (i) we get, for $m > \nu$, $\frac{1}{2^{m-2}} \mid S_2 - S_1 \mid < \epsilon$.

$\therefore \quad \mid S_{m+p} - S_m \mid < \epsilon$ for $m > \nu$.

$\therefore$ By general principle of convergence $\{S_n\}$ is convergent.

Now as in the first method, $2S_{n+1} + S_n = 2S_2 + S_1$

$\therefore$ Making $n \to \infty$, $2l + l = 2S_2 + S_1$ where $l = \lim_{n \to \infty} S_n$.

$$\therefore \quad l = \tfrac{1}{3}(2S_2 + S_1).$$

7. *If* s_1, s_2 *are positive and* $s_{n+2} = \sqrt{s_{n+1}\, s_n}$, *then prove that the sequence* $\{s_n\}$ *is convergent and find its limit.*

(यदि s_1, s_2 धनात्मक हों और $s_{n+2} = \sqrt{s_{n+1} \cdot s_n}$ तो सिद्ध करें कि अनुक्रम $\{s_n\}$ अभिसारी है एवं उसकी सीमा निकालें।)

(B U 1958 H; '60 H, '68 H; P U '53 H, '67 H;
Bh U '63 H, '67 H, '70 H)

Solution. Let $s_1 > s_2$.

Then, from the given relation, $s_{n+2} = \sqrt{s_{n+1} s_n}$.

We have, by putting $n = 1$, $s_3 = \sqrt{s_2 s_1}$.

This shows that s_3 is the geometric mean between s_1 and s_2.

$$\therefore \quad s_1 > s_3 > s_2, \text{ as } s_1 > s_2. \qquad \ldots (1)$$

Again, when $n = 2$, $s_4 = \sqrt{s_3 s_2}$; $\therefore$ s_2, s_4, s_3 are in G. P.

$$\Rightarrow \quad s_3 > s_4 > s_2, \text{ as, from (1), } s_3 > s_2. \qquad \ldots (2)$$

From (1) and (2), we get

$$s_1 > s_3 > s_4 > s_2. \qquad \ldots (3)$$

Again, when $n = 3$, $s_5 = \sqrt{s_4 s_3}$; $\therefore$ s_3, s_5, s_4 are in G. P.

$$\Rightarrow \quad s_3 > s_5 > s_4, \text{ as, from (2), } s_3 > s_4. \qquad \ldots (4)$$

From (3) and (4), we get $s_1 > s_3 > s_5 > s_4 > s_2$.

Repeating this process, we get

$$s_1 > s_3 > s_5 > s_7 > \ldots > s_8 > s_6 > s_4 > s_2.$$

From this we conclude that the sequence $\{s_{2n+1}\}$ is monotonic decreasing, whose every term is greater than s_2.

Also the sequence $\{s_{2n}\}$ is monotonic increasing whose every term is less than s_1.

Hence both the sequences are monotonic and bounded and therefore they are both convergent.

Similarly, if $s_2 > s_1$, then we can prove that the sequence $\{s_{2n+1}\}$ is monotonic increasing whereas the sequence $\{s_{2n}\}$ is monotonic decreasing and both of them are bounded and hence convergent.

Let $\lim\limits_{n\to\infty} s_{2n+1} = l$; then $\lim\limits_{n\to\infty} s_{2n-1} = l$

and $\lim\limits_{n\to\infty} s_{2n} = m$.

$$\because \quad s_{2n+1} = \sqrt{s_{2n} . s_{2n-1}}$$

$$\Rightarrow \quad \lim_{n\to\infty} s_{2n+1} = \sqrt{\lim_{n\to\infty} s_{2n} . \lim_{n\to\infty} s_{2n-1}}$$

$$\Rightarrow \quad l = \sqrt{l . m}$$

$$\Rightarrow \quad l = m,$$

$$\therefore \quad \lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} = \lim_{n\to\infty} s_{2n-1} = \ldots = l.$$

$$\because \quad s_{2n+1}^2 = s_{2n}s_{2n-1},\ s_{2n}^2 = s_{2n-1}\, s_{2n-2}, \text{ etc,}$$

$$\begin{aligned}\therefore \quad s^2_{2n+1}s_{2n} &= s_{2n}^2\, s_{2n-1}\\ &= s_{2n-1}^2 s_{2n-2}\\ &= s_{2n-2}^2 s_{2n-3}\\ &= \ldots\ldots\ldots\\ &= s_2^2 s_1\end{aligned}$$

$$\Rightarrow \quad \lim_{n\to\infty} s^2_{2n+1} \lim_{n\to\infty} s_{2n} = s_2^2 . s_1$$

$$\Rightarrow \quad l^2 . l = s_2^2 . s_1$$

$$\Rightarrow \quad l = (s_2^2 . s_1)^{\frac{1}{3}}.$$ This is the required limit.

8. ***A sequence $\{a_n\}$ is defined by $a_{n+1} = \sqrt{k + a_n}$, where $k > 0, a_1 > 0$.***

Prove that the sequence $\{a_n\}$ is monotonic and converges to the positive root of the equation $x^2 - x - k = 0$.

(अनुमक $\{a_n\}$ इस प्रकार परिभाषित है—

$$a_{n+1} = \sqrt{k + a_n} \text{ जहाँ } k > 0, a_1 > 0.$$

सिद्ध करें कि अनुक्रम $\{a_n\}$ एकदिष्ट है एवं यह समीकरण $x^2 - x - k = 0$ के धनमूल की ओर अभिसारी है।) (Bh U 1965 H; Mith. U. '85; P U '59 H, '63 H, '71 H, '74 H; B U '62 H; R U '62 H)

Solution. We have $a_{n+1} = \sqrt{k + a_n}$

$$\Rightarrow \quad a^2_{n+1} = k + a_n \qquad \text{.. (1)}$$

Replacing n by $n-1$, $a_n^2 = k + a_{n-1}$

$$\therefore \quad a^2_{n+1} - a_n^2 = k + a_n - k - a_{n-1}$$

$$\Rightarrow \quad a^2_{n+1} - a_n^2 = a_n - a_{n-1}.$$

Hence $a_{n+1} >$ or $< a_n$

according as $a_n >$ or $< a_{n-1}$,

that is, according as $a_{n-1} >$ or $< a_{n-2}$,

.. ..

that is, according as $a_2 >$ or $< a_1$.

Hence the sequence $\{a_n)$ is monotonic increasing or decreasing according as $a_2 >$ or $< a_1$.

The given equation is $x^2 - x - k = 0$.

The product of the roots of this equation is $-k$, which is negative, as from question $k > 0$.

This shows that one root must be positive and the other negative.

Let the roots be α and $-\beta$ where $\alpha > 0$ and $\beta > 0$.

$\therefore \quad x^2 - x - k = (x - \alpha)(x + \beta)$

$\therefore \quad a_1^2 - a_1 - k = (a_1 - \alpha)(a_1 + \beta)$

$\Rightarrow \quad a_1^2 - (a_1 + k) = (a_1 - \alpha)(a_1 + \beta)$

$\Rightarrow \quad a_1^2 - a_2^2 = (a_1 - \alpha)(a_1 + \beta)$,

putting $n = 1$ in (1).

$\because$ The sequence $\{a_n\}$ is monotonic increasing or decreasing according as $a_2 >$ or $< a_1$; that is, $a_2^2 >$ or $< a_1^2$,

that is, $a_1^2 - a_2^2 <$ or > 0,

that is, $(a_1 - \alpha)(a_1 + \beta) <$ or > 0,

that is, $a_1 <$ or $> \alpha$, as $a_1 > 0$ and $\beta > 0$.

If it is monotonic increasing,

then $a_{n+1} > a_n$ for all n

$\Rightarrow \quad a^2_{n+1} > a_n^2$

$\Rightarrow \quad k + a_n > a_n^2$

$\Rightarrow \quad a_n^2 - a_n - k < 0$

$\Rightarrow \quad (a_n - \alpha)(a_n + \beta) < 0$

$\Rightarrow \quad a_n < \alpha$ for all n, since $a_n + \beta > 0$.

Hence in this case the sequence is bounded and hence convergent.

Again if it is monotonic decreasing then $a_{n+1} < a_n$ for all n

$\Rightarrow \quad a^2_{n+1} < a_n^2$

$\Rightarrow \quad k + a_n < a_n^2$

$\Rightarrow \quad a_n^3 - a_n - k > 0$

$\Rightarrow \quad (a_n - \alpha)(a_n + \beta) > 0$

$\Rightarrow \quad a_n > \alpha.$

Hence the sequence is bounded below and therefore it must be convergent.

Thus in both cases the limit exists.

Let the limit be l, that is, let $\lim\limits_{n \to \infty} a_n = l.$

Now $a^2_{n+1} = k + a_n$

$\Rightarrow \quad \lim\limits_{n \to \infty} a^2_{n+1} = k + \lim\limits_{n \to \infty} a_n$

$\Rightarrow \quad l^2 = k + l$

$\Rightarrow \quad l^2 - l - k = 0.$

Since l is positive, therefore $\{a_n\}$ converges to the positive root of $x^2 - x - k = 0$.

9. *If* u_1, v_1 *are given positive unequal numbers and* $u_n = \frac{1}{2}(u_{n-1} + v_{n-1})$, $v_n = \sqrt{u_{n-1}v_{n-1}}$, *for* $n \geqslant 2$, *prove that each of the sequences* $\{u_n\}$ *and* $\{v_n\}$ *tends to a limit, which is the same for both.*

(यदि u_1, v_1 प्रदत्त असमान धन-संख्याएँ हों और $u_n = \frac{1}{2}(u_{n-1} + v_{n-1})$, $v_n = \sqrt{u_{n-1}v_{n-1}}$ जहां $n \geqslant 2$, तो सिद्ध करें कि अनुक्रम $\{u_n\}$ और $\{v_n\}$ में से प्रत्येक समान सीमा की ओर प्रवृत्त होता है।)

(R U 1967 H; P U '51 H, '55 H)

Solution. We have $u_n = \frac{1}{2}(u_{n-1} + v_{n-1})$; putting $n = 2$, we get $u_2 = \frac{1}{2}(u_1 + v_1)$.

Therefore u_2 is the arithmetic mean between u_1 and v_1.

Again, we have $v_n = \sqrt{u_{n-1}v_{n-1}}$;

putting $n = 2$, we get $v_2 = \sqrt{u_1 v_1}$.

Therefore v_2 is the geometric mean between u_1 and v_1.

But A.M. > G.M.

$$\therefore \quad u_2 > v_2. \qquad \ldots (1)$$

CASE I. Let $u_1 > v_1$.

Since u_2 is the A.M. between u_1 and v_1, therefore

$$u_1 > u_2 > v_1. \qquad \ldots (2)$$

Again $$v_2 = \sqrt{u_1 v_1} > \sqrt{v_1 . v_1}, \text{ because } u_1 > v_1$$

$$\Rightarrow \quad v_2 > v_1. \qquad \ldots (3)$$

From (1), (2) and (3), we get $u_1 > u_2 > v_2 > v_1$.

Repeating this process, we get

$$u_1 > u_2 > u_3 > \ldots > v_3 > v_2 > v_1.$$

Hence the sequence $\{u_n\}$ is monotonic decreasing and is bounded by u_1, v_1 and therefore it must be convergent.

Again the sequence $\{v_n\}$ is monotonic increasing and is bounded by u_1, v_1 and therefore it must be convergent.

CASE II. Similarly, when $u_1 < v_1$ we can prove that both the sequences $\{u_n\}$ and $\{v_n\}$ are convergent.

Hence in both the cases they must tend to a limit.

Let $\lim_{n\to\infty} u_n = l = \lim_{n\to\infty} u_{n-1}$ and $\lim_{n\to\infty} v_n = m = \lim_{n\to\infty} v_{n-1}$.

Now $$u_n = \tfrac{1}{2}(u_{n-1} + v_{n-1})$$

$$\Rightarrow \lim_{n\to\infty} u_n = \tfrac{1}{2}\left(\lim_{n\to\infty} u_{n-1} + \lim_{n\to\infty} v_{n-1}\right)$$

$$\Rightarrow \quad 2l = l + m$$

$$\Rightarrow \quad l = m.$$

Hence both the sequences tend to the same limit.

10. *Show that* $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$.

(Bh U 1966 H; B U '58 H, '62 H;
P U '52 H, '63 H, '70 H)

Solution. Let $u_n = n^{\frac{1}{n}}$, n^{th} root being non-negative and real.

As in problem 3, we can prove that $\left(1+\dfrac{1}{n}\right)^n < 3$ for all n.

$$\therefore\quad \left(1+\frac{1}{n}\right)^n < 3 \leqslant n \text{ if } n \geqslant 3.$$

$$\therefore\quad n > \left(1+\frac{1}{n}\right)^n \quad \forall\, n \geqslant 3$$

or $$n^{n+1} > (1+n)^n \quad \forall\, n \geqslant 3$$

or $$n^{\frac{1}{n}} > (1+n)^{\frac{1}{n+1}} \quad \forall\, n \geqslant 3. \qquad \ldots \quad (1)$$

$$\therefore\quad u_n > u_{n+1} \;\forall\, n \geqslant 3.$$

$\therefore$ $\{u_n\}$ is monotonic decreasing for $n \geqslant 3$.

Also $u_3 = 3^{\frac{1}{3}}$ and $u_n > 0 \;\forall\, n \in N$.

$u_1 = 1$, $u_2 = 2^{\frac{1}{2}} > 1$ and $3^{\frac{1}{3}} > 2^{\frac{1}{2}}$.

Thus $0 < u_n < 3^{\frac{1}{3}} \;\forall\, n \in N$.

$\therefore$ $\{u_n\}$ is bounded and monotonic decreasing.

$\therefore$ It is convergent to its greatest lower bound.

By (1), $u_n = n^{\frac{1}{n}} > (1+n)^{\frac{1}{n+1}} > 1 \;\forall\, n \geqslant 3$.

Also $u_1 = 1$, $u_2 = 2^{\frac{1}{2}} > 1$.

$\therefore$ 1 is the greatest lower bound. So $u_n \to 1$ as $n \to \infty$.

$$\therefore\quad \lim_{n\to\infty} n^{\frac{1}{n}} = 1.$$

Alternative Method.

Let $u_n = n^{\frac{1}{n}}$, n^{th} root being taken non-negative and real.

As above by (1), $u_n = n^{\frac{1}{n}} > (1+n)^{\frac{1}{n+1}} > 1$ for $n \geqslant 3$.

So we take $u_n = 1 + h$, where $h > 0$.

$$\therefore \quad n = u_n{}^n = (1+h)^n$$

$$= 1 + nh + \frac{n(n-1)}{\lfloor 2} h^2 + \ldots + h^n$$

$\Rightarrow \quad n > \frac{1}{2} n(n-1)h^2$ i.e. one term of the above series of positive terms.

$$\Rightarrow \quad h^2 < \frac{2}{n-1}$$

$$\Rightarrow \quad 0 < h < \sqrt{\frac{2}{n-1}}.$$

If ϵ be any positive number, however small, then

$$\sqrt{\frac{2}{n-1}} < \epsilon, \text{ when } n > 1 + \frac{2}{\epsilon^2}.$$

Hence if m be any integer greater than $1 + \frac{2}{\epsilon^2}$, we find that, for $n \geqslant m$,

$$-\epsilon < 0 < h < \sqrt{\frac{2}{n-1}} < \epsilon$$

$$\Rightarrow \quad |h| < \epsilon$$

$$\Rightarrow \quad \lim_{n \to \infty} h = 0$$

$$\Rightarrow \quad \lim_{n \to \infty} (1+h) = 1$$

$$\Rightarrow \quad \lim_{n \to \infty} n^{\frac{1}{n}} = 1.$$

11. *Prove that*

$$\lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} - \log n\right) = \gamma,$$

where $0 < \gamma < 1$.

Solution. Let

$$u_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{n} - \log n.$$

Then $u_{n-1} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} - \log(n-1)$.

$$\therefore \quad u_n - u_{n-1} = \frac{1}{n} - \log n + \log(n-1)$$

$$= \frac{1}{n} + \log\frac{n-1}{n}$$

$$= \frac{1}{n} + \log\left(1 - \frac{1}{n}\right)$$

$$= \frac{1}{n} - \frac{1}{n} - \frac{1}{2n^2} - \frac{1}{3n^3} - \ldots \text{to } \infty$$

$$= -\left(\frac{1}{2n^2} + \frac{1}{3n^3} + \ldots \text{to } \infty\right)$$

$\Rightarrow \quad u_n - u_{n-1} < 0$

$\Rightarrow \quad u_n < u_{n-1}$.

Hence the given sequence $\{u_n\}$ is monotonic decreasing.

Again $u_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{n} - \log n$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{n} - \log\left(\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \ldots \frac{n}{n-1}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{n} - \log\frac{2}{1} - \log\frac{3}{2} - \log\frac{4}{3} - \ldots - \log\frac{n}{n-1}$$

$$= (1 - \log 2) + \left(\frac{1}{2} - \log\frac{3}{2}\right) + \left(\frac{1}{3} - \log\frac{4}{3}\right) + \ldots + \left(\frac{1}{n-1} - \log\frac{n}{n-1}\right) + \frac{1}{n}.$$

Since $x > \log(1 + x)$, $\therefore$ $1 > \log 2$, $\frac{1}{2} > \log\frac{3}{2}$, etc.

$\therefore$ $u_n > 0$, for all n. .. (1)

Hence the given sequence $\{u_n\}$ is bounded below.

Thus $\{u_n\}$ must be convergent.

Hence it must tend to a limit.

Let $\lim\limits_{n \to \infty} u_n = \gamma$.

$$\text{Again } u_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log\left(\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \ldots \frac{n}{n-1}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \log\frac{1}{2} + \log\frac{2}{3} + \log\frac{3}{4} + \ldots + \log\frac{n-1}{n}$$

$$= 1 + \left(\frac{1}{2} + \log\frac{1}{2}\right) + \left(\frac{1}{3} + \log\frac{2}{3}\right) + \ldots + \left(\frac{1}{n} + \log\frac{n-1}{n}\right)$$

$$= 1 + \left\{\frac{1}{2} + \log\left(1 - \frac{1}{2}\right)\right\} + \left\{\frac{1}{3} + \log\left(1 - \frac{1}{3}\right)\right\} + \ldots + \left\{\frac{1}{n} + \log\left(1 - \frac{1}{n}\right)\right\}.$$

$\because \; x + \log(1 - x) < 0, \; \therefore \; \frac{1}{2} + \log(1 - \frac{1}{2}) < 0$, etc.

$\therefore \; u_n < 1$, for all n. .. (2)

From (1) and (2), we get

$$0 < u_n < 1$$

$$\Rightarrow \; 0 < \lim_{n \to \infty} u_n < 1$$

$$\Rightarrow \; 0 < \gamma < 1.$$

12. *Evaluate* (मान निकालें) :

$$\lim_{n\to\infty}\left[\left(\frac{2}{1}\right)^{1}\left(\frac{3}{2}\right)^{2}\left(\frac{4}{3}\right)^{3}\cdots\left(\frac{n+1}{n}\right)^{n}\right]^{\frac{1}{n}}$$

Solution. Let the given limit be denoted by y.

$$\text{Then } y=\lim_{n\to\infty}\left[(1+1)^{1}\left(1+\frac{1}{2}\right)^{2}\left(1+\frac{1}{3}\right)^{3}\cdots\left(1+\frac{1}{n}\right)^{n}\right]^{\frac{1}{n}}$$

$$\Rightarrow\quad \log y=\lim_{n\to\infty}\frac{1}{n}\left[\log(1+1)+2\log\left(1+\frac{1}{2}\right)+3\log\left(1+\frac{1}{3}\right)+\cdots+n\log\left(1+\frac{1}{n}\right)\right].$$

$$\text{Let } u_n=n\log\left(1+\frac{1}{n}\right)$$

$$=n\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\cdots\right)$$

$$=1-\frac{1}{2n}+\frac{1}{3n^2}-\cdots$$

$$\Rightarrow\quad \lim_{n\to\infty}u_n=1.$$

Hence, by Cauchy's first theorem on limit, we have

$$\lim_{n\to\infty}\frac{u_1+u_2+u_3+\cdots+u_n}{n}=1.$$

Hence $\log y=1=\log e$

$\Rightarrow \quad y=e.$

Hence the given limit $= e$.

Problem-Set 2

1. If $u_n=\dfrac{n^2+1}{2n^2+5}$, find m such that

$$\left|u_n-\frac{1}{2}\right|<\epsilon, \text{ for all } n\geqslant m.$$

2. Show that the sequence $\{u_n\}$ where $u_n = \frac{1}{n} \sin \frac{n\pi}{2}$ is convergent. (P U 1966)

3. If $u_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n}$, prove that $\frac{1}{2} < u_n < 1$ and that u_n tends to a limit as $n \to \infty$.
(P U 1955 H, '75, '78; M. U. '85; B. U. '85; Mith. U. '85)

4. Find the limit of the sequence $\{a_n\}$ where

$$a_n = \frac{1}{n} + \frac{1}{2n} + \ldots + \frac{1}{n^2}. \qquad \text{(P U 1980)}$$

5. Show that the sequence $\{a_n\}$ is not convergent when

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}. \qquad \text{(P U 1979)}$$

6. Show that the sequence

$$\tfrac{2}{1}, \tfrac{3}{2}, \tfrac{5}{4}, \tfrac{9}{8}, \tfrac{17}{16}, \ldots$$

is convergent and find its limit. (B U 1972)

7. If $u_n = \frac{1.3.5.\ldots(2n-1)}{3.6.9.\ldots 3n}$, then show that the sequence $\{u_n\}$ is monotonic and hence deduce its limits. (P U 1967)

8. (i) If $a_{n+1} = \sqrt{1 + a_n}$ and $a_1 > 0$, prove that the sequence $\{a_n\}$ tends to a limit and find the limit. (P U 1954 H)

(ii) Show that the sequence $\{a_n\}$ defined by

$$a_n = \sqrt{5 + a_{n-1}} \text{ and } a_1 = 1$$

tends to a definite and finite limit. (P U 1965 H)

(iii) Prove that the sequence

$$\sqrt{7}, \sqrt{7 + \sqrt{7}}, \sqrt{7 + \sqrt{7 + \sqrt{7}}}, \ldots$$

converges to the positive root of $x^2 - x - 7 = 0$.
(P U 1961; B U '53 H)

(iv) Show that the sequence

$$\sqrt{2}, \sqrt{2 + \sqrt{2}}, \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \ldots$$

converges to 2. (R U 1968 H)

9. (i) If $a_2 > a_1$ and $a_n^2 = 2a_{n-1}$ then show that
$a_n \to 2$ as $n \to \infty$. (P U 1976)

(ii) Prove that the sequence
$\sqrt{2}, \sqrt{(2\sqrt{2})}, \sqrt{\{2\sqrt{(2\sqrt{2})}\}}, \ldots$ converges to 2.
(P. U. 1985)

10. If $0 < s_1 < s_2$ and $s_n = \dfrac{2s_{n-1}s_{n-2}}{s_{n-1}+s_{n-2}}$

(i.e. s_n is the harmonic mean between s_{n-1} and s_{n-2}),

show that $s_n \to \dfrac{3s_1s_2}{2s_1+s_2}$. (P U 1957 H)

11. If a_1 and b_1 are positive and

$$a_{n+1} = \frac{1}{2}(a_n + b_n),\ b_{n+1} = \frac{2a_n b_n}{a_n + b_n},\ \text{for } n \geqslant 1$$

then prove that $\{a_n\}$ and $\{b_n\}$ are monotonic sequence which converge to the same limit $\sqrt{a_1 b_1}$.

12. Prove that $\lim\limits_{n\to\infty} \dfrac{(\lfloor n)^{\frac{1}{n}}}{n} = \dfrac{1}{e}$.

13. If $a_n = \dfrac{k}{1+a_{n-1}}$ where $a_n > 0\ \forall\ n \in N$, show that the sequence $\{a_n\}$ tends to a definite limit l which is the positive root of the equation $x^2 + x = k$.

14. If $a_n \to a$ and $b_n \to b$ show that

$$\frac{a_1 b_n + a_2 b_{n-1} + \ldots + a_n b_1}{n} \to ab.$$

15. If $\{u_n\}$ be a sequence of positive terms defined by

$$a_1 = k > 0,\ a_{n+1} = \frac{3+2a_n}{2+a_n} \quad \text{for } n \geqslant 1$$

then show that the sequence converges to a limit independent of k and find its limit.

16. If $\{a_n\}$ be a sequence such that $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = l$ then prove that

(i) $\{a_n\}$ diverges to $+\infty$ if $l > 1$;
(ii) $\{a_n\}$ converges to 0 if $|l| < 1$.

17. Show that

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\frac{1}{\sqrt{n^2+3}}+\ldots+\frac{1}{\sqrt{n^2+n}}\right]$$

$= 1.$

(Hindi Version)

1. यदि $u_n = \dfrac{n^2+1}{2n^2+5}$ तो एक m ज्ञात करें ताकि प्रत्येक $n \geqslant m$ के लिए

$$| u_n - \tfrac{1}{2} | < \epsilon.$$

2. सिद्ध करें कि अनुक्रम $\{u_n\}$ अभिसारी है जहाँ $u_n = \dfrac{1}{n} \sin \dfrac{n\pi}{2}$.

3. यदि $u_n = \dfrac{1}{n+1} + \dfrac{1}{n+2} + \dfrac{1}{n+3} + \ldots + \dfrac{1}{2n}$ तो सिद्ध करें कि $\frac{1}{2} < u_n < 1$ और u_n एक सीमा की ओर प्रवृत्त होता है जबकि $n \to \infty$.

4. अनुक्रम $\{a_n\}$ की सीमा निकालें जहाँ

$$a_n = \frac{1}{n} + \frac{1}{2n} + \ldots + \frac{1}{n^2}.$$

5. सिद्ध करें कि अनुक्रम $\{a_n\}$ अभिसारी नहीं है जहाँ

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}.$$

6. सिद्ध करें कि अनुक्रम

$$\frac{2}{1}, \frac{3}{2}, \frac{5}{4}, \frac{9}{8}, \frac{17}{16}, \ldots$$

अभिसारी है और इसकी सीमा निकालें ।

7. यदि $u_n = \dfrac{1.3.5\ldots.(2n-1)}{3.6.9\ldots..3n}$, सिद्ध करें कि अनुक्रम $\{u_n\}$ एकदिष्ट है और फिर इसकी सीमा निकालें ।

8. (i) यदि $a_{n+1} = \sqrt{1+a_n}$ और $a_1 > 0$, तो सिद्ध करें कि अनुक्रम $\{a_n\}$ एक सीमा की ओर प्रवृत्त होता है एवं वह सीमा निकालें ।

(ii) दिखलायें कि अनुक्रम $\{a_n\}$ एक सुनिश्चित एवं परिमित सीमा की ओर प्रवृत्त होता है जबकि अनुक्रम इस प्रकार परिभाषित है—

$$a_n = \sqrt{5 + a_{n-1}} \text{ और } a_1 = 1.$$

(iii) सिद्ध करें कि अनुक्रम

$$\sqrt{7},\ \sqrt{7+\sqrt{7}},\ \sqrt{7+\sqrt{7+\sqrt{7}}},\ \ldots\ldots$$

समीकरण $x^2-x-7=0$ के धनमूल की ओर अभिसारी है।

(iv) सिद्ध करें कि अनुक्रम

$$\sqrt{2},\ \sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}},\ \ldots.$$

2 की ओर अभिसारी है।

9. (i) यदि $a_2 > a_1$ और $a_n{}^2 = 2a_{n-1}$ तो दिखलायें कि

$$a_n \to 2 \text{ जब } n \to \infty.$$

(ii) सिद्ध करें कि अनुक्रम

$$\sqrt{2},\ \sqrt{(2\sqrt{2})},\ \sqrt{\{2\sqrt{(2\sqrt{2})}\}},\ldots.$$

2 की ओर अभिसारी है।

10. यदि $0<S_1<S_2$ और $S_n=\dfrac{2S_{n-1}\cdot S_{n-2}}{S_{n-1}+S_{n-2}}$ (अर्थात् S_n, S_{n-1} और S_{n-2} का हरात्मक माध्य है) तो दिखलायें कि $S_n \to \dfrac{3S_1S_2}{2S_1+S_2}$.

11. यदि a_1 और b_1 धनात्मक हों और

$$a_{n+1}=\frac{1}{2}(a_n+b_n),\ b_{n+1}=\frac{2a_n\,b_n}{a_n+b_n} \text{ जबकि } n \geqslant 1$$

तो सिद्ध करें कि $\{a_n\}$ और $\{b_n\}$ एकदिष्ट अनुक्रम हैं जो सीमा $\sqrt{a_1b_1}$ की ओर अभिसारी है।

12. सिद्ध करें कि $\displaystyle\lim_{n\to\infty}\frac{(\lfloor n)^{\frac{1}{n}}}{n}=\frac{1}{e}$.

13. यदि $a_n=\dfrac{k}{1+a_{n-1}}$ जहाँ $a_n>0\ \forall\ n\in N$, तो सिद्ध करें कि अनुक्रम $\{a_n\}$ एक सुनिश्चित सीमा l की ओर प्रवृत्त होता है जो समीकरण $x^2+x=k$ का धन मूल है।

14. यदि $a_n \to a$ और $b_n \to b$ तो सिद्ध करें कि

$$\frac{a_1b_n+a_2b_{n-1}+\ldots+a_nb_1}{n}\to ab.$$

15. यदि $\{u_n\}$ धन पदों का अनुक्रम हो जो इस प्रकार परिभाषित है कि

$a_1 = k > 0,\ a_{n+1} = \dfrac{3 + 2a_n}{2 + a_n}$ जहाँ $n \geqslant 1$ तो सिद्ध करें कि अनुक्रम, k से मुक्त एक सीमा की ओर प्रवृत्त होता है और वह सीमा निकालें।

16. यदि अनुक्रम $\{a_n\}$ के लिए $\lim\limits_{n \to \infty} \dfrac{a_{n+1}}{a_n} = l$ तो सिद्ध करें कि

(i) $\{a_n\}$, $+\infty$ की ओर अपसारी है जबकि $l > 1$;

(ii) $\{a_n\}$, 0 की ओर अभिसारी है जबकि $|l| < 1$.

17. सिद्ध करें कि

$$\lim_{n \to \infty} \left[\frac{1}{\sqrt{n^2 + 1}} + \frac{1}{\sqrt{n^2 + 2}} + \frac{1}{\sqrt{n^2 + 3}} + \ldots + \frac{1}{\sqrt{n^2 + n}} \right] = 1.$$

CHAPTER
3

CONVERGENCE OF SERIES

3.1. Definitions

Series. An expression of the form

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots,$$

in which the successive terms are terms of a sequence $\{u_n\}$ is called a *series*; if the series terminates at some assigned term i.e. it contains first few terms of a sequence, it is called a *finite series*; if the number of terms of the series is unlimited, it is called an *infinite series.* (B U 1962 H, '64 H; R U '62 H)

If the terms of a series are all non-negative i.e. $u_n \geqslant 0$ then the sereis is a ***series of positive terms***. In general, terms of a series may be positive or negative or zero.

Examples. The series $1 + 2 + 3 + 4 + \ldots + n$

and $1 + r + r^3 + r^3 + \ldots + r^n$

are finite series, while the series

$$1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \ldots + \frac{1}{x^n} + \ldots \text{to } \infty$$

and $$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots \text{ to } \infty.$$

are infinite series.

In this chapter only infinite series will be dealt with.

In short, the infinite series will be written as series.

The series $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ to ∞ is denoted by $\sum\limits_{n=1}^{\infty} u_n$ or by Σu_n and the sum of its first n terms by S_n.

So $$S_n = u_1 + u_2 + u_3 + \ldots + u_n.$$

As u_n is a function of n, a series is also denoted as $\Sigma f(n)$.

3.2. Sequence of Partial Sums of a Series; Convergence of Series

Let $u_1 + u_2 + \ldots + u_n + \ldots$ be an infinite series. If S_n denotes the sum of the first n terms of this series, so that $S_n = u_1 + u_2 + \ldots + u_n$, then the sequence $\{S_n\}$ is called the sequence of partial sums of the given series.

Convergent Series· If $\{S_n\}$ tends to a finite and definite limit S, then S is defined to be the sum to infinity of the series and the series is said to be convergent to the sum S.

Thus S is defined as $\lim\limits_{n\to\infty} S_n = S$ or $S_n \to S$, as $n \to \infty$.

Divergent Series. If $\{S_n\}$ tends to $+\infty$ or to $-\infty$, as $n \to \infty$, then the series is said to be divergent.

In other words, the series is divergent if having given any positive number $\triangle$ whatsoever, we can find a definite number m such that $S_n > \triangle$, when $n \geqslant m$.

Oscillatory Series. If $\{S_n\}$ tends to no definite limit whether finite or infinite as $n \to \infty$, then the series is said to oscillate. We say that the series oscillates finitely or infinitely according as S_n oscillates between finite limits or between $+\infty$ and $-\infty$.

3.3. Some Direct Consequences of the Definition of Convergence of Series

Theorem 3.1. The alteration, addition or omission of a finite number of terms of a series has no effect on its being convergent, divergent or oscillatory.

Theorem 3.2. Multiplication of all the terms of a series by a fixed non-zero number has no effect on its being convergent, divergent or oscillatory i.e. $\sum ku_n$ is convergent, divergent or oscillatory according as $\sum u_n$ is convergent, divergent or oscillatory.

Note. It is important to note here that the sum to infinity of a convergent series is essentially a limit and is not a sum in the sense described in the definition of addition. We are, therefore, not justified in assuming that the sum of an infinite series is unaltered by changing the order of the terms or by the introduction or removal of brackets.

In fact, changes of this kind may alter the sum or they may transform a convergent series into one which diverges or oscillates.

For example, the series $(1-1)+(1-1)+(1-1)+\ldots$ is convergent and its sum is zero, but the series $1-1+1-1+\ldots$, which is obtained by removing the brackets of the original series, oscillates.

Examples. (i) Let us consider the series

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\text{to }\infty.$$

Here $S_n = 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^{n-1}}$

$$=\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}=2\left(1-\frac{1}{2^n}\right)=2-\frac{1}{2^{n-1}}.$$

Now, when $n\to\infty$, $\frac{1}{2^{n-1}}\to 0$.

$\therefore \lim\limits_{n\to\infty} S_n = 2.$

Thus S_n tends to a finite limit as $n\to\infty$.

Hence the series $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots$ to ∞ is convergent and its sum is 2, which is really the limit sum.

(ii) Let us consider the series

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ldots+\frac{1}{n(n+1)}+\ldots\text{to }\infty.$$

Here $u_n=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$.

Putting $n = 1, 2, 3, 4, \ldots, n$ successively, we get

$$u_1 = 1 - \tfrac{1}{2}$$
$$u_2 = \tfrac{1}{2} - \tfrac{1}{3}$$
$$u_3 = \tfrac{1}{3} - \tfrac{1}{4}$$
$$\cdots \quad \cdots \quad \cdots$$
$$\cdots \quad \cdots \quad \cdots$$
$$u_n = \frac{1}{n} - \frac{1}{n+1}$$

Adding, we get $S_n = 1 - \dfrac{1}{n+1}$.

$\therefore \lim\limits_{n \to \infty} S_n = 1.$

The series $\dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + \ldots$ to ∞ is therefore convergent and its sum is unity (which is really the limit sum).

(iii) Let us consider the series

$$1 + 2 + 3 + 4 + \ldots + n + \ldots \text{to } \infty.$$

Here $S_n = 1 + 2 + 3 + 4 + \ldots + n$

$$= \frac{n(n+1)}{2}.$$

$\therefore \lim\limits_{n \to \infty} S_n = \infty.$

Hence the series $1 + 2 + 3 + \ldots$ to ∞ is divergent.

(iv) Let us consider the series

$$1 + 4 + 7 + 10 + \ldots \text{to } \infty.$$

Here $S_n = 1 + 4 + 7 + 10 + \ldots$ to n terms, which is an A.P.

$= \dfrac{n}{2}\{2.1 + (n-1).3\}$, from the formula

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

$$= \frac{n}{2}(3n - 1).$$

$\therefore$ When $n \to \infty$, $S_n \to \infty$.

Hence the series $1 + 4 + 7 + 10 + \ldots$ to ∞ is divergent.

(v) Let us consider the series

$$1 - 1 + 1 - 1 + 1 - 1 + \dots \text{to } \infty.$$

Here $S_n = 1 - 1 + 1 - 1 + 1 - 1 + \dots$ to n terms

$= 0$ or 1, according as n is even or odd.

Hence $\lim\limits_{n \to \infty} S_n = 0$ or 1.

Thus the series $1 - 1 + 1 - 1 + \dots$ to ∞ is oscillatory and oscillates finitely.

(vi) Let us consider the series

$$1 - 2 + 3 - 4 + 5 - 6 + \dots \text{to } \infty.$$

Here $S_n = 1 - 2 + 3 - 4 + 5 - 6 + \dots$ to n terms

$= -\dfrac{n}{2}$ or $\dfrac{n+1}{2}$, according as n is even or odd.

$\therefore \ S_n \to -\infty$ or $+\infty$ as $n \to \infty$.

Hence the series $1 - 2 + 3 - 4 + 5 - 6 + \dots$ to ∞ oscillates infinitely.

3.4. Convergence of a Geometric Series

The geometric series $1 + r + r^2 + r^3 + \dots$ *to* ∞ *is convergent when* $|r| < 1$; *the series is divergent when* $r \geqslant 1$; *the series oscillates finitely when* $r = -1$ *and oscillates infinitely when* $r < -1$.

Proof. The sum S_n of the first n terms of the given series

$$= \frac{1 - r^n}{1 - r};\ r \neq 1.$$

(i) If $|r| < 1$, $r^n \to 0$ as $n \to \infty$.

$$\therefore \quad S_n \to \frac{1}{1 - r} \text{ as } n \to \infty.$$

Hence the given geometric series is convergent when

$$|r| < 1.$$

(ii) If $r > 1$, $r^n \to +\infty$ as $n \to \infty$.

$$\therefore \qquad S_n \to +\infty \text{ as } n \to \infty.$$

If $r = 1$, $S_n = 1 + 1 + 1 + \dots$ to n terms $= n$.

$$\therefore \qquad S_n \to +\infty \text{ as } n \to \infty.$$

Hence the geometric series is divergent and it diverges to $+\infty$ when $r \geqslant 1$.

(iii) If $r = -1$, $S_n = 1 - 1 + 1 - 1 + \ldots$ to n terms

$= 0$ or 1, according as n is even or odd.

$\therefore \qquad S_n \to 0$ or 1 as $n \to \infty$.

Hence the geometric series $\sum r^{n-1}$ oscillates finitely when $r = -1$.

(iv) If $r < -1$, $1 - r^n \to +\infty$ or $-\infty$ according as n is odd or even.

Hence $S_n \to +\infty$ or $-\infty$ when $n \to \infty$.

Thus the series $\sum r^{n-1}$ oscillates infinitely.

3.5. Cauchy's General Principle of Convergence for Series'

Theorem 3.3. *A necessary and sufficient condition for a series* $\sum u_n$ *to converge is that to each* $\epsilon > 0$, *there exists a positive integer* m *such that*

$$| u_{m+1} + u_{m+2} + \ldots + u_n | < \epsilon, \text{ whenever } n > m.$$

Proof. The series $\sum u_n$ converges if and only if the sequence $\{S_n\}$ of its partial sums converges.

By Cauchy's general principle of convergence for sequences, a necessary and sufficient condition for the convergence of $\{S_n\}$ is that given any $\epsilon > 0$, there exists a positive integer m such that $| S_n - S_m | < \epsilon$, whenever $n > m$,

i.e., $| u_{m+1} + u_{m+2} + \ldots + u_n | < \epsilon$ whenever $n > m$.

Hence the required result.

Theorem 3.4. *If an infinite series* $\sum u_n$ *is convergent, then*

$$u_n \to 0 \text{ as } n \to \infty.$$

(P U 1971 H; B U '54 H; R U '62 H, '70 H)

Proof. Let $S_n = u_1 + u_2 + \ldots + u_n$.

Since $\sum u_n$ is convergent, therefore, let $S_n \to S$, a definite quantity, as $n \to \infty$.

Also $\qquad S_{n-1} \to S$ as $n \to \infty$.

Now, $S_n = u_1 + u_2 + \ldots + u_{n-1} + u_n$

and $S_{n-1} = u_1 + u_2 + \ldots + u_{n-1}$

$$\therefore \quad S_n = S_{n-1} + u_n;$$

i.e., $$u_n = S_n - S_{n-1}.$$

$$\therefore \lim_{n \to \infty} u_n = \lim_{n \to \infty} (S_n - S_{n-1}) = \lim_{n \to \infty} S_n - \lim_{n \to \infty} S_{n-1}$$

$$= S - S = 0.$$

Hence $u_n \to 0$ as $n \to \infty$.

Note. It is important to note here that $u_n \to 0$, as $n \to \infty$, is a necessary condition for the convergence of the series $\sum u_n$ but *it is not a sufficient condition.*

Let us illustrate it by an example.

For example, consider the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots \text{to } \infty.$$

Here $u_n = \dfrac{1}{n}$.

$$\therefore \quad \lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{n} = 0.$$

But the series is divergent.

For, $$\tfrac{1}{3} + \tfrac{1}{4} > \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2},$$

$$\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} > \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} = \tfrac{1}{2}, \text{ and so on.}$$

$$\therefore \quad 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots \text{to } \infty$$

$$> (1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \ldots \text{to } \infty) \to \infty$$

and is therefore divergent.

Hence $u_n \to 0$ as $n \to \infty$, but the series $\sum u_n$ is not convergent.

Example. Test for convergence $\sum \dfrac{n(n+1)}{(n+2)^2}$. (R U 1970 H)

Solution. Let the given series be denoted by $\sum u_n$.

Then $u_n = \frac{n(n+1)}{(n+2)^2} = \frac{1+\frac{1}{n}}{\left(1+\frac{2}{n}\right)^2}$

$\Rightarrow \lim\limits_{n \to \infty} u_n = 1 \neq 0.$

Hence the given series is divergent.

3.6. Convergence of Series of Positive Terms

In the analysis of convergence of a series, the series of positive terms hold an important place. Obviously for a series of positive terms, the sum of n terms S_n goes on increasing as more and more terms are added up. However it does not guarantee that the sum of infinite terms will exceed any prescribed finite number. It may happen that the increase in the sum goes on decreasing as more and more terms are added up and ultimately this increase becomes negligible i.e. to say the sum gets closer and closer to a definite number.

By the definition of a convergent series, it becomes essential to find the partial sum S_n and then determine its limit as n tends to infinity. But to find S_n for every series is not possible. Under the circumstances we are inclined to establish such results which can determine whether a series is convergent or not without bothering to determine the partial sum S_n of the series.

Theorem 3.5. *A series* $\sum u_n$ *of positive terms is either convergent or divergent.*

Proof. Let $S_n = u_1 + u_2 + u_3 + \ldots + u_n$.

So $\{S_n\}$ is the sequence of partial sums of the series $\sum u_n$.

As $S_{n+1} = u_1 + u_2 + u_3 + \ldots + u_n + u_{n+1}$

$= S_n + u_{n+1}$ and $u_{n+1} \geqslant 0$

(being series of positive terms),

$\therefore \qquad S_{n+1} \geqslant S_n$ for all $n \in N$.

$\therefore$ $\{S_n\}$ is monotonic increasing which may be bounded or unbounded above.

If $\{S_n\}$ is bounded above, by a theorem on monotonic sequence, we get $\{S_n\}$ is convergent and consequently $\sum u_n$ is convergent.

If $\{S_n\}$ is unbounded above, we know $S_n \to \infty$ as $n \to \infty$.

$\therefore \quad \sum u_n$ is divergent.

Note. A series of non-negative terms cannot oscillate.

Theorem 3.6. *If $\sum u_n$ and $\sum v_n$ are two series of positive terms, convergent to the sums s and s' respectively then the series $\sum(u_n + v_n)$ and $\sum(u_n - v_n)$ are also convergent to the sums $s + s'$ and $s - s'$ respectively.*

Proof. Let the sum of n term of $\sum(u_n + v_n)$ be denoted by V_n.

$$\begin{aligned} \text{Then } V_n &= (u_1 + v_1) + (u_2 + v_2) + \ldots + (u_n + v_n) \\ &= (u_1 + u_2 + \ldots + u_n) + (v_1 + v_2 + \ldots + v_n) \\ &= s_n + s_n'. \end{aligned}$$

As $\sum u_n$ and $\sum v_n$ are convergent to s and s', $s_n \to s$ and $s_n' \to s'$.

$\therefore \quad V_n \to s + s'$ i.e. $\sum(u_n + v_n)$ is convergent to the sum $s + s'$.

Similarly $\sum(u_n - v_n)$ is convergent to the sum $s - s'$.

Theorem 3.7. *A series $\sum u_n$ of positive terms converges if and only if there exists a finite number K such that*

$$u_1 + u_2 + \ldots + u_n < K \textit{ for all } n.$$

Proof. Let us first suppose that there exists a number K such that $u_1 + u_2 + \ldots + u_n < K$, for all n.

Then the sequence $\{S_n\}$ of partial sums of the series $\sum u_n$ is bounded above.

Also, since $u_n > 0$ for all n, therefore the sequence $\{S_n\}$ is monotonic increasing as $S_{n+1} - S_n = u_{n+1} > 0$ for all $n \geqslant 1$.

Since every monotonic increasing sequence bounded above converges, therefore $\{S_n\}$ and consequently $\sum u_n$ converges.

Conversely, let us suppose that $\sum u_n$ converges. Then by definition, the sequence $\{S_n\}$ of partial sums of $\sum u_n$ converges.

Since every convergent sequence is bounded, therefore $\{S_n\}$ is bounded, i.e., there exist numbers k and K such that

$$k < S_n < K, \text{ for all } n.$$

From the second part of the above inequalities we get $u_1 + u_2 + \ldots + u_n < K$, for all n.

Note. If no such finite number can be found, the series will be divergent. For then $S_n > \triangle$ where $\triangle$ is arbitrary.

Theorem 3.8. *The terms of an infinite series of positive terms can be arranged in groups, without altering their order, in any way we like without affecting the convergence or divergence of the series.*

Proof. Let $\sum u_n$ be a series of positive terms and let the terms be arranged in groups without changing their order. Let the sum of the terms in the n^{th} group be denoted by v_n. So the new series is $\sum v_n$.

Let $\quad S_n = u_1 + u_2 + \dots + u_n$

and $\quad t_n = v_1 + v_2 + \dots + v_n$.

Then for every n we can find a positive number p, so that

$$S_n \leqslant t_n < S_{n+p}.$$

If $\sum u_n$ is convergent, then $\lim S_n = \lim S_{n+p} = S$.

Therefore $\lim t_n = S$.

Hence it follows that if $\sum u_n$ converges to a sum S, so does $\sum v_n$.

If $\sum u_n$ diverges, then as $t_n \geqslant S_n \to \infty$, so $\sum v_n$ also diverges.

Similarly it can be proved that $\sum u_n$ converges or diverges according as $\sum v_n$ converges or diverges.

Theorem 3.9. *If the terms of a convergent series of positive terms are re-arranged, the series remains convergent and its sum is unaltered.*

Proof. Let $\sum u_n$ be a convergent series of positive terms and let the terms of $\sum u_n$ be re-arranged in any manner. Let the new series, so formed, be denoted by $\sum v_n$. Then for every u there is v and for every v there is *an* u.

Let $\quad S_n = u_1 + u_2 + \dots + u_n$

and $\quad t_n = v_1 + v_2 + \dots + v_n$.

Let us suppose that the first n terms of $\sum u_n$ are among the first m terms of $\sum v_n$; and that the first m terms of $\sum v_n$ are among the first $(n + p)$ terms of $\sum u_n$.

Then $n \leqslant m \leqslant n + p$, where m tends to infinity with n.

Also, $S_n \leqslant t_m \leqslant S_{n+p}$.

Since $\sum u_n$ is a convergent series, therefore

$$\lim S_n = \lim S_{n+p} = S.$$

Hence $\lim t_m = S$.

Thus $\sum v_n$ is convergent and has the same sum as $\sum u_n$.

Note. This proof fails if $\sum u_n$ is broken up into two (or any finite number of) infinite series.

Theorem 3.10. (Auxiliary Series)

The infinite series $\sum \frac{1}{n^p}$ i.e.,

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\ldots \text{ to } \infty$$

is (*i*) *convergent if* $p > 1$
and (*ii*) *divergent if* $p \leqslant 1$. (M U 1979;
R U '73H; Mith.U '85; P U '85)

Proof. Let us denote the given series by S. Since S is a series of positive terms, its convergence or divergence is not affected by grouping the terms in brackets in any way we please.

CASE 1. *Let* $p > 1$.

Group the terms as follows :

$$\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)+\ldots \text{to } \infty.$$

Clearly $\frac{1}{1^p}=1$,

$$\frac{1}{2^p}+\frac{1}{3^p}<\frac{1}{2^p}+\frac{1}{2^p}=\frac{2}{2^p},$$

$$\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}<\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}=\frac{4}{4^p},$$

Similarly $\frac{1}{8^p}+\frac{1}{9^p}+\ldots+\frac{1}{15^p}<\frac{8}{8^p}$, and so on.

Adding the corresponding sides of the above relations, we have $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots \text{to } \infty$

$$< 1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \ldots \text{to } \infty$$

i.e., $S < \dfrac{1}{1 - \dfrac{2}{2^p}}$, sum to infinity of a series in G.P.

Since $p > 1$, therefore the right hand side of the above inequality is finite.

Hence S is convergent.

CASE 2. *Let* $p = 1$. Then the given series becomes

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} + \ldots \text{to } \infty.$$

Group the terms thus :

$$1 + \tfrac{1}{2} + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}) + \ \ldots \text{to } \infty.$$

Obviously $1 + \frac{1}{2} = 1 + \frac{1}{2}$,

$$\tfrac{1}{3} + \tfrac{1}{4} > \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{2}{4} = \tfrac{1}{2},$$

$$\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} > \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} = \tfrac{4}{8} = \tfrac{1}{2},$$

....

....

Adding the corresponding sides of the above relations, we obtain $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots \text{to } \infty$

$$> 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \ldots \text{to } \infty.$$

But the series $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots \text{to } \infty$ is divergent. Hence the given series is also divergent.

CASE 3. *Let* $p < 1$.

Group the terms as follows :

$$\frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + \ldots \text{to } \infty.$$

$$\text{Obviously,}\quad \frac{1}{1^p} = 1,$$

$$\frac{1}{2^p} + \frac{1}{3^p} > \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{1}{2},$$

$$\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2},$$

$$\frac{1}{8^p} + \frac{1}{9^p} + \frac{1}{10^p} + \ldots + \frac{1}{15^p}$$

$$> \tfrac{1}{16} + \tfrac{1}{16} + \tfrac{1}{16} + \ldots + \tfrac{1}{16} = \tfrac{8}{16} = \tfrac{1}{2},$$

$$\ldots\ldots\ldots \qquad \ldots\ldots\ldots \qquad \ldots\ldots$$

Adding the corresponding sides of the above relations, we find that

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots \text{to } \infty > 1 + \frac{1}{2} + \frac{1}{2} + \ldots \text{to } \infty.$$

But the latter series is divergent. Hence the given series is divergent.

Hence the series $\Sigma \dfrac{1}{n^p}$ is always divergent except in the case when p is positive and greater than unity.

3.7. Pringsheim's Theorem or Abel's Theorem

(P U 1959 H, '63 H, '67 H, '70 H; B U '58 H, '62 H, '64 H, '68 H; M U '64 H, '69 H; R U '63 H, '73 H; Bh U '63 H)

[The theorem was discovered by Abel but forgotten, and rediscovered by Pringsheim.]

Theorem 3.11. *If* Σu_n *is a convergent series of positive decreasing terms, then* $\lim\limits_{n \to \infty} (n\, u_n) = 0$.

Proof. Since Σu_n is a series of positive decreasing terms, therefore

$$\begin{aligned} u_{n+1} &\geqslant u_{2n} \\ u_{n+2} &\geqslant u_{2n} \\ u_{n+3} &\geqslant u_{2n} \\ \ldots&\ldots\ldots \\ u_{2n} &= u_{2n} \end{aligned}$$

Adding, we get $u_{n+1} + u_{n+2} + u_{n+3} + \ldots + u_{2n} \geqslant n\, u_{2n}$

Since $\sum u_n$ is convergent, by the general condition for convergence,

$$u_{n+1} + u_{n+2} + u_{n+3} + \ldots + u_{2n} \to 0 \text{ as } n \to \infty.$$

$$\therefore \lim_{n \to \infty} (n\, u_{2n}) = 0$$

$$\Rightarrow \lim_{n \to \infty} (2n\, u_{2n}) = 0. \quad .. \quad (1)$$

Hence Pringsheim's theorem is proved when n is even.

$$\text{Again } \lim_{n \to \infty} \left(\frac{2n+1}{2n}\right) = \lim_{n \to \infty} \left(1 + \frac{1}{2n}\right) = 1. \quad .. \quad (2)$$

Now, $u_{2n+1} \leqslant u_{2n}$

$$\therefore \quad (2n+1)u_{2n+1} \leqslant \frac{2n+1}{2n}.(2n\, u_{2n})$$

$$\Rightarrow \quad (2n+1)u_{2n+1} \leqslant \left(1 + \frac{1}{2n}\right)(2n\, u_{2n})$$

$$\Rightarrow \lim_{n \to \infty} [(2n+1)u_{2n+1}] \leqslant \lim_{n \to \infty}\left(1 + \frac{1}{2n}\right) [\lim_{n \to \infty} (2n\, u_{2n})]$$

$$\Rightarrow \lim_{n \to \infty} [(2n+1)u_{2n+1}] = 0, \text{ from (1) and (2).}$$

Hence Pringsheim's theorem is proved when n is odd.

Hence $\lim_{n \to \infty} (n\, u_n) = 0$.

Example. Use Pringsheim's theorem to prove that the series $\sum \frac{1}{n}$ is divergent.

Solution. Here $u_n = \frac{1}{n}$.

$$\therefore \quad n\, u_n = n . \frac{1}{n} = 1.$$

$$\therefore \quad \lim_{n \to \infty} (n\, u_n) = 1 \neq 0.$$

Hence, by Pringsheim's theorem, $\sum u_n$ i.e., $\sum \frac{1}{n}$ cannot be convergent and since it is a series of positive terms, it must be divergent.

3.9. Some Remarks on Pringsheim's Theorem

(i) To show, by an example, that Pringsheim's theorem is not true if the condition, that u_n decreases as n increases, is omitted. (Bh. U. 1963 H)

Solution. Let us consider the series

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}+\frac{1}{9}+\frac{1}{10^2}+\ldots.$$

Hence, the nth term $u_n = \dfrac{1}{n}$ or $\dfrac{1}{n^2}$, according as n is or is not a perfect square.

Re-arrange the series (of positive terms) in the form

$$\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}+\frac{1}{10^2}+\ldots + \left(1+\frac{1}{4}+\frac{1}{9}+\ldots\right).$$

Each of these series is convergent.

Hence the series under consideration is convergent.

But since $n\,u_n = 1$, whenever n is a perfect square, it is not true that $n\,u_n \to 0$.

Hence, in this case Pringsheim's theorem is not true.

(ii) To show, by an example, that the converse of Pringsheim's theorem is not necessarily true. (B. U. 1962 H)

Solution. Let us consider the series

$$1+\tfrac{1}{2}\cdot\tfrac{1}{2}+\tfrac{1}{3}\left(\tfrac{1}{3}+\tfrac{1}{4}\right)+\tfrac{1}{4}\left(\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}\right)+\ldots. \quad (1)$$

Since $1+\tfrac{1}{2}\cdot\tfrac{1}{2}+\tfrac{1}{3}\left(\tfrac{1}{3}+\tfrac{1}{4}\right)+\tfrac{1}{4}\left(\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}\right)+\ldots.$ is greater than $1+\tfrac{1}{2}\cdot\tfrac{1}{2}+\tfrac{1}{3}\cdot\tfrac{1}{2}+\tfrac{1}{4}\cdot\tfrac{1}{2}+\ldots$, therefore the series (1) is divergent as $1+\tfrac{1}{2}\cdot\tfrac{1}{2}+\tfrac{1}{3}\cdot\tfrac{1}{2}+\tfrac{1}{4}\cdot\tfrac{1}{2}+\ldots$ is divergent.

But it is easy to see that the terms of the series (1), that is, $1+\tfrac{1}{2}\cdot\tfrac{1}{2}+\tfrac{1}{3}\cdot\tfrac{1}{3}+\tfrac{1}{3}\cdot\tfrac{1}{4}+\tfrac{1}{4}\cdot\tfrac{1}{5}+\tfrac{1}{4}\cdot\tfrac{1}{6}+\ldots.$ satisfy the condition that $n\,u_n \to 0$.

Hence the converse of Pringsheim's theorem is not true in this case.

3.10. Elementary Tests for Convergence of Series of Positive terms

Theorem 3.12 (Comparison Test)

(B U 1955 H; R U '70 H)

FORM I. (i) *If* $u_n \leqslant kv_n$, *for all values of* n, *where* $n > 0$ *and* k *is a fixed positive number, and* $\sum v_n$ *is convergent, then* $\sum u_n$ *is also convergent.*

(ii) *If* $u_n \geqslant kv_n$, *for all values of* n, *where* $n > 0$ *and* k *is a fixed positive number, and* $\sum v_n$ *is divergent, then* $\sum u_n$ *is also divergent.*

Proof. Let $S_n = u_1 + u_2 + u_3 + \ldots + u_n$

and $\sigma_n = v_1 + v_2 + v_3 + \ldots + v_n$.

(i) From the inequality $u_n \leqslant kv_n$, for all n, we have

$$u_1 + u_2 + u_3 + \ldots + u_n \leqslant k(v_1 + v_2 + v_3 + \ldots + v_n),$$

i.e. $S_n \leqslant k\sigma_n$, for all n. .. (1)

Since $\sum v_n$ is convergent, therefore there must exist a positive number A, such that

$$v_1 + v_2 + v_3 + \ldots + v_n < A, \text{ for all } n,$$

i.e. $\sigma_n < A$, for all n. .. (2)

From (1) and (2), we obtain

$$S_n < kA, \text{ for all } n.$$

The sequence $\{S_n\}$ of partial sums of the series $\sum u_n$ is thus monotonic increasing and bounded above by kA and hence $\sum u_n$ converges.

(ii) From the inequality $u_n \geqslant kv_n$ for all n, we get

$$u_1 + u_2 + u_3 + \ldots + u_n \geqslant k(v_1 + v_2 + v_3 + \ldots + v_n),$$

i.e. $S_n \geqslant k\sigma_n$ for all n.

Since $\sum v_n$ is divergent, therefore

$$\sigma_n \to \infty \text{ as } n \to \infty.$$

$$\therefore \quad S_n \to \infty \text{ as } n \to \infty.$$

Hence $\sum u_n$ is divergent.

Note. It is sufficient if the conditions in the above test are satisfied for all values of $n \geqslant$ some fixed integer m.

FORM II. *If* $\lim_{n \to \infty} \frac{u_n}{v_n}$ = *a finite and non-zero number* l, *then* $\sum u_n$ *and* $\sum v_n$ *are either both convergent or both divergent.*

Proof. Since $\frac{u_n}{v_n} \to l$ as $n \to \infty$, therefore by the definition of limit it follows that corresponding to $\epsilon > 0$ there exists a positive integer m such that

$$\left| \frac{u_n}{v_n} - l \right| < \epsilon, \text{ for all } n \geqslant m.$$

Let $0 < \epsilon < l$, then $0 < l - \epsilon < \frac{u_n}{v_n} < l + \epsilon$, for $n \geqslant m$.

Since $v_n > 0$ for all n, therefore multiplying the above inequalities throughout by v_n, we have

$$0 < (l - \epsilon)\, v_n < u_n < (l + \epsilon)\, v_n, \text{ for } n \geqslant m.$$

Now if $\sum v_n$ is convergent, then $\sum (l + \epsilon)\, v_n$ is also convergent and therefore $\sum u_n$ is convergent, as $u_n < (l + \epsilon)\, v_n$. [from Form I (i)]

If, however, $\sum v_n$ is divergent, then $\sum (l - \epsilon)\, v_n$ is also divergent and therefore $\sum u_n$ is divergent as $u_n > (l - \epsilon)\, v_n$. [from Form I (ii)]

Note. (i) This test is also stated in the following form by combining (i) and (ii) of the form I.

If $a \leqslant \frac{u_n}{v_n} \leqslant b$ for all values of $n \geqslant m$, where a and b are positive constants, the two series $\sum u_n$ and $\sum v_n$ converge or diverge simultaneously.

(ii) A test for convergence of a series is nothing but a set of sufficient conditions under which the series is convergent or divergent.

3.11. Use of Comparison Test

This test is applicable for series of positive terms only.

For a given series $\sum u_n$ of positive terms, a series $\sum v_n$ of positive terms is to be selected whose convergence is known and

$\lim\limits_{n\to\infty} \dfrac{u_n}{v_n}$ is non-zero finite number. The efficiency in choosing Σv_n requires practice. However if

$$u_n = \frac{P(x)}{Q(x)} \text{ then choose}$$

$$v_n = \frac{\text{term containing highest power of } x \text{ in } P(x)}{\text{term containing highest power of } x \text{ in } Q(x)}$$

which will be finally of the form $\dfrac{1}{n^p}$. Thus $\Sigma \dfrac{1}{n^p}$ will act as the primary series (or auxiliary series) for testing the convergence of a series by comparison test.

WORKED OUT EXAMPLES

1. *Test the convergence of the series—*
श्रेणी के अभिसरण की जाँच करें—

$$1 + \frac{1+2}{1+2^2} + \frac{1+3}{1+3^2} + \ldots + \frac{1+n}{1+n^2} + \ldots + \text{ to } \infty.$$

(M U 1969)

or, Test the convergence of the series $\sum\limits_{n=1}^{\infty} \dfrac{1+n}{1+n^2}$.

Solution. Here the n^{th} term, $u_n = \dfrac{1+n}{1+n^2}$.

Let us compare the given series Σu_n with the auxiliary series Σv_n, where $v_n = \dfrac{n}{n^2} = \dfrac{1}{n}$.

Then
$$\frac{u_n}{v_n} = \frac{n+n^2}{1+n^2} = \frac{\frac{1}{n}+1}{\frac{1}{n^2}+1}.$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{n}+1}{\frac{1}{n^2}+1} = 1, \text{ which is finite and } \neq 0.$$

Hence, by comparison test, Σu_n and Σv_n are either both convergent or both divergent.

But $\Sigma v_n = \Sigma \frac{1}{n}$ i.e. $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$ to ∞

is divergent as $\Sigma \frac{1}{n^p}$ is divergent when $p = 1$.

Hence the given series is also divergent.

2. *Test the convergence of the series*

$$1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \frac{4^4}{5^5} + \ldots + \frac{(n-1)^{n-1}}{n^n} + \ldots \text{to } \infty.$$

(P U 1964; B U 57, '63; Roorkee Engg. '76; Mith. U. '82)

Solution. Here the n^{th} term, $u_u = \dfrac{(n-1)^{n-1}}{n^n}$.

Let us compare the given series Σu_n with the auxiliary series Σv_n, where

$$v_n = \frac{n^{n-1}}{n^n} = \frac{1}{n}.$$

Then $\dfrac{u_n}{v_n} = \dfrac{n(n-1)^{n-1}}{n^n} = \dfrac{n}{n-1} \cdot \dfrac{(n-1)^n}{n^n}$

$$= \frac{1}{1 - \frac{1}{n}} \cdot \left(1 - \frac{1}{n}\right)^n.$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \left[\frac{1}{1-\frac{1}{n}} \cdot \left(1 - \frac{1}{n}\right)^n\right] = e^{-1},$$

which is finite and non-zero

$$\left[\because \operatorname*{Lt}_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = e^{-1}\right].$$

So, by comparison test, Σu_n and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma \frac{1}{n}$, i.e., $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots$ to ∞ is divergent as $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots$ to ∞ is divergent, when

$$p = 1.$$

Hence $\sum u_n$ is also divergent.

3. (i) *Prove that the following series is convergent—*

सिद्ध करें कि निम्नलिखित श्रेणी अभिसारी है—

$$\frac{1.2}{3^2.4^2}+\frac{3.4}{5^2.6^2}+\frac{5.6}{7^2.8^2}+\ldots \qquad \text{(P U)}$$

Solution : Here

$$u_n = \frac{(2n-1).2n}{(2n+1)^2.(2n+2)^2} = \frac{1}{n^2}\cdot\frac{\left(2-\frac{1}{n}\right)\times 2}{\left(2+\frac{1}{n}\right)^2.4\left(1+\frac{1}{n}\right)^2}.$$

Put $\qquad v_n = \frac{1}{n^2};$

$$\therefore \ \underset{n\to\infty}{\text{Lt}}\frac{u_n}{v_n} = \underset{n\to\infty}{\text{Lt}}\frac{\left(2-\frac{1}{n}\right)}{\left(2+\frac{1}{n}\right)^2.2\left(1+\frac{1}{n}\right)^2} = \frac{1}{4}$$

which is finite and $\neq 0$.

Therefore $\sum u_n$ and $\sum v_n$ converge or diverge together.

But $\sum v_n = \sum\frac{1}{n^2}$ is convergent, as $p = 2 > 1$ in $\sum\frac{1}{n^p}$.

So $\sum u_n$ is also convergent.

3. (ii) *Test the convergence of the series—*

$$1+\frac{2^2}{3^2}+\frac{2^2}{3^2}\cdot\frac{4^2}{5^2}+\frac{2^2.4^2.6^2}{3^2.5^2.7^2}+\ldots \textit{to infinity.} \qquad \text{(P U)}$$

Solution. Here $u_{n+1} = u_n.\frac{(2n)^2}{(2n+1)^2}$.

Now, $\quad u_1 = 1;\ u_2 = \frac{2^2}{3^2} = \frac{4}{9} > \frac{3}{9} = \frac{1}{3};$

$$u_3 = u_2 \cdot \frac{4^2}{5^2} = u_2 \cdot \frac{16}{25} > \frac{1}{3} \cdot \frac{15}{25} = \frac{1}{5};$$

Similarly, let $u_m > \frac{1}{2m-1}$.

$$\therefore \quad u_{m+1} = u_m \cdot \frac{(2m)^2}{(2m+1)^2} > \frac{1}{2m-1} \cdot \frac{(2m)^2 - 1}{(2m+1)^2} = \frac{1}{2m+1}$$

or $u_{m+1} > \frac{1}{2(m+1)-1}$.

$\therefore \quad u_n > \frac{1}{2n-1}$ is true for all n.

Take $v_n = \frac{1}{2n-1}$; then $v_n > \frac{1}{2n}$ and so $\Sigma v_n > \frac{1}{2} \cdot \sum \frac{1}{n}$ and consequently Σv_n is divergent. Hence Σu_n is also divergent.

3. (iii) *Discuss the convergence of the series—*
श्रेणी के अभिसरण को व्याख्या करें—

$$\frac{1}{\sqrt{2}-1} + \frac{1}{\sqrt{3}-1} + \frac{1}{\sqrt{4}-1} + \ldots . \textit{ad inf.}$$

[P U 1949; M U '72, '80]

Solution. Here the n^{th} term, $u_n = \frac{1}{\sqrt{n+1}-1}$.

Let us compare the given series Σu_n with the auxiliary series Σv_n, where $v_n = \frac{1}{\sqrt{n}}$.

Then $\frac{u_n}{v_n} = \frac{\sqrt{n}}{\sqrt{n+1}-1} = \frac{1}{\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}}$.

$\therefore \lim\limits_{n\to\infty} \frac{u_n}{v_n} = \lim\limits_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}} - \frac{1}{\sqrt{n}}} = 1$, which is finite and non-zero.

So, by comparison test, Σu_n and Σv_n converge or diverge together.

But $\Sigma v_n = \sum \frac{1}{\sqrt{n}}$, i.e. $\frac{1}{1^{\frac{1}{2}}}+\frac{1}{2^{\frac{1}{2}}}+\frac{1}{3^{\frac{1}{2}}}+\ldots$ to ∞ is divergent as

$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots$ to ∞ is divergent when $p\,(=\frac{1}{2})<1$.

Hence $\sum u_n$ is also divergent.

4. *Test the convergence of the series whose general term is*—
श्रेणी के अभिसरण की जाँच करें जिसका सामान्य पद निम्नलिखित है—

$$\frac{\sqrt{n+1}-\sqrt{n}}{n}. \qquad \text{(P U 1951, '55)}$$

Solution. Here the n^{th} term,

$$u_n=\frac{\sqrt{n+1}-\sqrt{n}}{n}$$

$$=\frac{\sqrt{n+1}-\sqrt{n}}{n}\times\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$=\frac{n+1-n}{n(\sqrt{n+1}+\sqrt{n})}=\frac{1}{n(\sqrt{n+1}+\sqrt{n})}.$$

Let us compare the given series $\sum u_n$ with the auxiliary series $\sum v_n$, where $v_n=\frac{1}{n.\sqrt{n}}=\frac{1}{n^{\frac{3}{2}}}$.

Then $\frac{u_n}{v_n}=\frac{n^{\frac{3}{2}}}{n(\sqrt{n+1}+\sqrt{n})}=\frac{1}{\sqrt{1+\frac{1}{n}}+1}$.

$\therefore\ \lim\limits_{n\to\infty}\frac{u_n}{v_n}=\lim\limits_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1}=\frac{1}{2}$, which is finite and non-zero.

So, by comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

But $\sum v_n=\sum\frac{1}{n^{3/2}}$, i.e. $\frac{1}{1^{3/2}}+\frac{1}{2^{3/2}}+\frac{1}{3^{3/2}}+\cdots$ to ∞ is convergent as $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots$ to ∞ is convergent when $p\,(=\frac{3}{2})>1$.

Hence the given series is also convergent.

5. *Test the convergence of the series whose general term is*

$$\sqrt{n^4+1}-\sqrt{n^4-1}.$$

(Bh U 1962, '69; B U '85, '77; P U '69, '75, '81; M U '75)

Solution. Here the n^{th} term,

$$u_n=\sqrt{n^4+1}-\sqrt{n^4-1}$$

$$=\frac{\sqrt{n^4+1}-\sqrt{n^4-1}}{\sqrt{n^4+1}+\sqrt{n^4-1}}\times(\sqrt{n^4+1}+\sqrt{n^4-1})$$

$$=\frac{(n^4+1)-(n^4-1)}{\sqrt{n^4+1}+\sqrt{n^4-1}}=\frac{2}{\sqrt{n^4+1}+\sqrt{n^4-1}}.$$

Let us compare the given series Σu_n with the auxiliary series Σv_n, where $v_n=\frac{1}{\sqrt{n^4}}=\frac{1}{n^2}$.

Then $$\frac{u_n}{v_n}=\frac{2n^2}{\sqrt{n^4+1}+\sqrt{n^4-1}}=\frac{2}{\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}}.$$

$$\therefore\quad \lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{2}{\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}}$$

$$=\frac{2}{1+1}=\frac{2}{2}=1,$$

which is finite and non-zero.

So, by comparison test, Σu_n and Σv_n converge or diverge together.

But $\Sigma v_n=\Sigma\frac{1}{n^2}$, i.e. $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$ to ∞ is convergent as $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots$ to ∞ is convergent when $p(=2)>1$.

Hence Σu_n is also convergent.

6. *Test the convergence of the series* $\sum\limits_{n=1}^{\infty}\left(\sqrt[3]{n^3+1}-n\right)$.

(Bh U 1964; B U '72)

Solution. Here the n^{th} term, $u_n = \sqrt[3]{n^3+1}-n$

$$\text{i.e.}\quad u_n = n\left(1+\frac{1}{n^3}\right)^{\frac{1}{3}}-n$$

$$= n\left[1+\frac{1}{3}.\frac{1}{n^3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{1\cdot 2}.\frac{1}{n^6}+\ldots-1\right]$$

$$= \frac{1}{3}.\frac{1}{n^2}+\frac{\frac{1}{3}.\left(\frac{1}{3}-1\right)}{1.2}.\frac{1}{n^5}+\ldots$$

Let us compare the given series $\sum u_n$ with the auxiliary series $\sum v_n$, where $v_n = \frac{1}{n^2}$.

$$\text{Then}\quad \frac{u_n}{v_n} = \frac{1}{3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{1.2}.\frac{1}{n^3}+\ldots$$

$$\therefore\quad \lim_{n\to\infty}\frac{u_n}{v_n} = \frac{1}{3}\ \text{which is finite and non-zero.}$$

So, by comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

But $\sum v_n = \sum\frac{1}{n^2}$, i.e. $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$ to ∞ is convergent as $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots$ to ∞ is convergent when $p(=2)>1$.

Hence the given series is also convergent.

7. *Discuss the convergence of the series for* $p \in R$—

$$\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\frac{5}{4^p}+\ldots \textit{to } \infty.$$

(B U 1956, '59; P U '79)

Solution. Here the n^{th} term, $u_n = \dfrac{n+1}{n^p}$.

Let us compare the given series $\sum u_n$ with the auxiliary series $\sum v_n$, where $v_n = \dfrac{n}{n^p} = \dfrac{1}{n^{p-1}}$.

Then $\dfrac{u_n}{v_n} = \dfrac{(n+1)n^{p-1}}{n^p} = \dfrac{n+1}{n} = 1 + \dfrac{1}{n}$.

$\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right) = 1$, which is finite and non-zero.

So, by comparison test, the two series $\sum u_n$ and $\sum v_n$ converge or diverge simultaneously.

But $\sum v_n = \sum \dfrac{1}{n^{p-1}}$, i.e. $\dfrac{1}{1^{p-1}} + \dfrac{1}{2^{p-1}} + \dfrac{1}{3^{p-1}} + \ldots$ to ∞ is convergent or divergent according as

$$p - 1 > 1 \quad \text{or} \quad p - 1 \leqslant 1,$$

i.e.

$$p > 2 \quad \text{or} \quad p \leqslant 2.$$

Hence the given series is convergent or divergent according as $p > 2$ or $p \leqslant 2$.

8. *Test the convergence of the series*

$$\frac{1}{1+x} + \frac{x}{1+x^2} + \frac{x^2}{1+x^3} + \frac{x^3}{1+x^4} + \ldots \textit{to } \infty, \ (x > 0).$$

(BPSC 1977; M U '66)

Solution. Here the n^{th} term, $u_n = \dfrac{x^{n-1}}{1+x^n}$.

When $x < 1$, put $v_n = x^{n-1}$. Then $\dfrac{u_n}{v_n} = \dfrac{1}{1+x^n}$.

$\therefore \quad \lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \lim\limits_{n\to\infty} \dfrac{1}{1+x^n} = 1$, since $x^n \to 0$ as $n \to \infty$ when $x < 1$.

So, by comparison test,

$\sum u_n$ and $\sum v_n$ converge or diverge together.

But $\sum v_n = \sum x^{n-1}$ i.e. $1 + x + x^2 + x^3 + \ldots$ to ∞ is convergent, for it is a G.P. whose common ratio is less than 1.

When $x > 1$, $u_n = \dfrac{x^{n-1}}{1+x^n} = \dfrac{\dfrac{1}{x}}{\dfrac{1}{x^n}+1}$

$$\therefore \quad \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\dfrac{1}{x}}{\dfrac{1}{x^n}+1} = \frac{1}{x} \neq 0.$$

But $u_n \to 0$ as $n \to \infty$ is a necessary condition of convergency.

So, $\sum u_n$ is divergent when $x > 1$.

If $x = 1$, the given series becomes

$$\frac{1}{1+1} + \frac{1}{1+1} + \frac{1}{1+1} + \ldots$$

i.e. $\dfrac{1}{2} + \dfrac{1}{2} + \dfrac{1}{2} + \ldots$ ...to ∞, which is divergent.

Hence the given series is convergent when $x < 1$ and divergent when $x \geqslant 1$.

9. *Test for convergence of the series whose n^{th} term is* $\sin\left(\dfrac{1}{n}\right)$.

(R U 1970)

Solution. Here $\quad u_n = \sin\left(\dfrac{1}{n}\right)$.

Let $\quad v_n = \dfrac{1}{n}$.

$$\left\{ \text{Remember that } \sin\frac{1}{n} = \frac{1}{n} - \frac{1}{\lfloor 3}\cdot\frac{1}{n^3} + \ldots \right\}$$

Then $\quad \dfrac{u_n}{v_n} = \dfrac{\sin\dfrac{1}{n}}{\dfrac{1}{n}}$.

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \text{ where } \frac{1}{n} = \theta$$

$= 1$, which is finite and non-zero.

$\therefore$ By comparison test, $\sum u_n$ and $\sum v_n$ converge or diverge together.

But $\sum v_n$ i.e., $\sum \frac{1}{n}$ diverges as $p=1$ in $\sum \frac{1}{n^p}$. Therefore the given series i.e. $\sum u_n$ also diverges.

Problem-Set 3

Test the convergence of the following series—

1. (i) $\frac{2}{1}+\frac{7}{15}+\frac{12}{53}+\frac{17}{127}+\ldots+\frac{5n-3}{2n^3-1}+\ldots$ to ∞. (P U 1940)

(ii) $\frac{4}{2}+\frac{7}{9}+\frac{10}{28}+\frac{13}{65}+\ldots+\frac{3n+1}{n^3+1}+\ldots$ to ∞. (M U 1969)

(iii) $1+\frac{3}{5}+\frac{5}{13}+\frac{7}{25}+\frac{9}{41}+\ldots+\frac{2n-1}{2n^2-2n+1}+\ldots$ to ∞. (P U 1956)

(iv) $\sum\limits_{n=1}^{\infty} \frac{2n^2+1}{3n^3+5n^2+6}$. (Bh U 1966)

2. (i) $\sqrt{\frac{1}{2^3}}+\sqrt{\frac{2}{3^3}}+\sqrt{\frac{3}{4^3}}+\ldots$ to ∞. (R U 1967)

(ii) $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\ldots$ to ∞. (Roorkee Engg. 1976)

(iii) $\frac{2}{1}+\frac{3}{4}+\frac{4}{9}+\frac{5}{16}+\frac{6}{25}+\ldots$ ad inf.

(iv) $1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots$ to ∞. (P U 1961)

(v) $\frac{1}{2}+\frac{\sqrt{2}}{2+\sqrt{2}}+\frac{\sqrt{3}}{3+\sqrt{3}}+\frac{\sqrt{4}}{4+\sqrt{4}}+\ldots$ to ∞. (P U 1954, '68; R U '75)

(vi) $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots$ (Mith U 1980)

3. Test the convergence of the series whose general term (or n^{th} term) is as follows—

(i) $\frac{\sqrt{n}}{n^2+1}$. (R U 1962; P U '77)

(ii) $\frac{1}{\sqrt{1+n^2}+n}$. (P U 1936)

(iii) $\frac{\sqrt{n+1}+\sqrt{n}}{n^2}$ (iv) $\sqrt[3]{\frac{n^2+2}{2n^3+15}}$. (B U 1970)

4. Test for convergence the series whose general term is as follows—

(i) $\sqrt{n^2+1}-\sqrt{n^2-1}$. (P U 1933)

(ii) $\sqrt{n^3+1}-\sqrt{n^3}$. (B U 1962; Bh U '69; P U '77)

(iii) $\sqrt{n^5+1}-\sqrt{n^5}$. (P U 1965)

(iv) $\sqrt{n^3+1}-\sqrt{n^3-1}$.

(P U 1953; R U '64; B U '62; Mith U '78)

(v) $\sqrt{n^2+1}-n$. (P U 1985; M U '69; B U '78; Bh U '67; Mith U '76; R U '76)

5. Test the convergence of the series—

(i) $\Sigma\{\sqrt{n^4+1}-n^2\}$. (R U 1971; Mith. U. '85)

(ii) $\sum_{1}^{\infty}(\sqrt{n+1}-\sqrt{n})$. (B U 1976)

6. (i) Discuss the convergence of the series for $p \in R$—

$$\frac{1}{1^p}+\frac{1}{3^p}+\frac{1}{5^p}+\frac{1}{7^p}+\ldots\text{to }\infty.$$

(Bh U 1964; R U '63; P U '61)

(ii) Test for convergence the series whose n^{th} term is

$$\frac{\sqrt{n+1}-\sqrt{n}}{n^p}.$$ (R U 1966)

7. Examine the convergence of the series whose n^{th} term is

(i) $\dfrac{\cos^2 nx}{(2n+1)^2}$, (ii) $\dfrac{1}{n} \sin \dfrac{1}{n}$, (iii) $\tan^{-1} \dfrac{1}{n}$.

Theorem 3.13. D' Alembert's Test (or Ratio Test) (M. U. 1985)

FORM I. (i) *An infinite series of positive terms is convergent if from and after some fixed term the ratio of each term to the preceding term is numerically less than some quantity which is itself numerically less than unity.*

(ii) *The series is divergent if the above ratio is greater than or equal to unity.*

Proof. (i) Let the infinite series beginning from the fixed term be denoted by

$$u_1 + u_2 + u_3 + u_4 + \ldots \text{to } \infty$$

and let $\dfrac{u_2}{u_1} < r, \dfrac{u_3}{u_2} < r, \dfrac{u_4}{u_3} < r, \ldots$, where $r < 1$.

Then $u_1 + u_2 + u_3 + u_4 + \ldots + \text{to } \infty$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \frac{u_4}{u_1} + \ldots \text{to } \infty\right)$$

i.e. $u_1 + u_2 + u_3 + \ldots \text{to } \infty$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}.\frac{u_2}{u_1} + \frac{u_4}{u_3}.\frac{u_3}{u_2}.\frac{u_2}{u_1} + \ldots \text{to } \infty\right)$$

i.e. $\sum u_n < u_1(1 + r + r^2 + r^3 + \ldots \text{to } \infty)$

i.e. $\sum u_n < u_1. \dfrac{1}{1-r}$,

by sum of infinite geometric series as $r < 1$.

Clearly $\dfrac{u_1}{1-r}$ is finite as u_1 is a fixed term and $r < 1$.

Hence $\sum u_n$ is convergent.

(ii) Let the series beginning from the fixed term be

$$u_1 + u_2 + u_3 + \ldots \text{to } \infty.$$

and let $\dfrac{u_2}{u_1} \geqslant 1, \dfrac{u_3}{u_2} \geqslant 1, \dfrac{u_4}{u_3} \geqslant 1, \ldots$

i.e. $u_2 \geqslant u_1$; $u_3 \geqslant u_2 \geqslant u_1$; $u_4 \geqslant u_3 \geqslant u_2 \geqslant u_1$ and so on.

Then $S_n = u_1 + u_2 + u_3 + u_4 + \dots + u_n$,

i.e. $S_n \geqslant u_1 + u_1 + u_1 + u_1 + \dots$ to n terms

i.e. $S_n \geqslant nu_1 \to \infty$ as $n \to \infty$.

Hence $\sum u_n$ is divergent.

D'Alembert's Ratio test can be stated in the following form ·

FORM II. (*Limit form*) *Let* $\sum u_n$ *be an infinite series of positive terms such that* $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = l$.

Then the series (*i*) *converges if* $l < 1$;

(*ii*) *diverges if* $l > 1$;

(*iii*) *may converge or diverge, i.e. the test fails if* $l = 1$.

(Bh U 1976; P U '56 H)

Proof. Let $l < 1$.

Then we can choose a positive number ϵ such that

$$l + \epsilon < 1.$$

Let $l + \epsilon = k$, so that $k < 1$.

Since $\dfrac{u_{n+1}}{u_n} \to l$ as $n \to \infty$, therefore, by definition of limit, there exists a positive integer m such that

$$l - \epsilon < \frac{u_{n+1}}{u_n} < l + \epsilon, \text{ for all } n \geqslant m.$$

Since $u_n > 0$, multiplying the above inequalities throughout by u_n, we obtain

$$(l - \epsilon)u_n < u_{n+1} < (l + \epsilon)u_n, \text{ for all } n \geqslant m. \quad .. \quad (1)$$

Taking the second inequality $u_{n+1} < (l + \epsilon)u_n$,

i.e $u_{n+1} < ku_n$, for $n \geqslant m$, we get

$$\begin{aligned} &u_{m+1} < ku_m \\ &u_{m+2} < ku_{m+1} < k^2u_m \\ &u_{m+3} < ku_{m+2} < k^3u_m \\ &\dots\dots \quad \dots\dots \quad \dots\dots \\ &\dots\dots \quad \dots\dots \quad \dots\dots \\ &u_{m+r} < ku_{m+r-1} < k^ru_m. \end{aligned}$$

$\therefore \; u_{m+1} + u_{m+2} + u_{m+3} + \dots < ku_m(1 + k + k^2 + \dots)$ which is a geometric series with common ratio $k < 1$.

But since $ku_m(1 + k + k^2 + \ldots)$ is convergent as $0 < k < 1$, we get $u_{m+1} + u_{m+2} + u_{m+3} + \ldots$ is also convergent and hence $\sum u_n$ is convergent.

Suppose next that $l > 1$. Then we can choose a positive number ϵ such that $l - \epsilon > 1$.

Let $l - \epsilon = \lambda$, so that $\lambda > 1$.

Now taking the first inequality of (1), $(l - \epsilon)u_n < u_{n+1}$,

i.e. $u_{n+1} > \lambda u_n$ for $n \geqslant m$, we get

$$\begin{aligned} u_{m+1} &> \lambda\, u_m \\ u_{m+2} &> \lambda\, u_{m+1} > \lambda^2 u_m \\ u_{m+3} &> \lambda\, u_{m+2} > \lambda^3 u_m \\ \ldots\ldots &\quad \ldots\ldots \quad \ldots\ldots \\ \ldots\ldots &\quad \ldots\ldots \quad \ldots\ldots \\ u_{m+r} &> \lambda\, u_{m+r-1} > \lambda^r u_m. \end{aligned}$$

$$\therefore \quad u_{m+1} + u_{m+2} + u_{m+3} + \ldots > \lambda\, u_m(1 + \lambda + \lambda^2 + \ldots).$$

But the series $\lambda\, u_m(1 + \lambda + \lambda^2 + \ldots)$ is geometric with common ratio $\lambda > 1$; so it is divergent.

Therefore, $u_{m+1} + u_{m+2} + u_{m+3} + \ldots$is also divergent and hence $\sum u_n$ is divergent.

If $l = 1$, the above test fails, so the series may be convergent or divergent depending upon the nature of the series.

Note. To keep uniformity with other ratio tests, D'Alembert's test is also remembered by taking $\frac{u_n}{u_{n+1}}$ in place of $\frac{u_{n+1}}{u_n}$ in the above form. Thus it is stated as follows—

A series $\sum u_n$ of positive terms such that $\lim\limits_{n \to \infty} \frac{u_n}{u_{n+1}} = l$, is convergent if $l > 1$ and divergent if $l < 1$.

Nothing can be said by the test if $l = 1$.

In problems usually this form is used.

Theorem 3.14. Cauchy's Root Test.

(P U 1955 H, '59 H; Mith. U. '85; P. U. '85)

Let $\sum u_n$ *be an infinite series of positive terms and let*

$$\lim_{n\to\infty} u_n^{\frac{1}{n}} = l.$$

Then the series $\sum u_n$ *(i) converges* *if* $l < 1$;

(ii) diverges *if* $l > 1$;

(iii) no conclusion if $l = 1$.

Proof. (Rigorous) Since $u_n^{\frac{1}{n}} \to l$ as $n \to \infty$, therefore, from the definition of limit, it follows that there exists a positive integer m such that

$$l - \epsilon < u_n^{\frac{1}{n}} < l + \epsilon \quad \ldots \quad (1),$$

for all $n \geqslant m$ and $\epsilon > 0$.

Let $l < 1$. Then we can choose ϵ such that $l + \epsilon < 1$.

From the second inequality of (1), we have

$$u_n^{\frac{1}{n}} < l + \epsilon, \text{ for all } n \geqslant m,$$

i.e. $$u_n < (l + \epsilon)^n, \text{ for all } n \geqslant m.$$

Now $\sum (l + \epsilon)^n$ converges, as it is a geometric series with common ratio $l + \epsilon < 1$; therefore, $\sum u_n$ also converges.

Again let $l > 1$. Then we can choose ϵ such that

$$l - \epsilon > 1.$$

From the first inequality of (1), we obtain

$$u_n^{\frac{1}{n}} > l - \epsilon, \text{ for all } n \geqslant m,$$

i.e. $$u_n > (l - \epsilon)^n, \text{ for all } n \geqslant m.$$

But $\sum (l - \epsilon)^n$ diverges, as it is a geometric series with common ratio $l - \epsilon > 1$; therefore $\sum u_n$ also diverges.

This test fails when $\lim\limits_{n\to\infty} u_n^{\frac{1}{n}} = 1$.

Alternative Proof (Elementary) :

CASE I. Here $u_n^{\frac{1}{n}} = r$, where $r < 1$

$\Rightarrow \quad u_n = r^n$

$$\Rightarrow \quad u_1 + u_2 + u_3 + \ldots + u_n + \ldots \text{to } \infty$$
$$= r + r^2 + r^3 + \ldots + r^n + \ldots \text{to } \infty$$

$\Rightarrow \quad \sum u_n = \frac{r}{1-r}$, which is finite, as $r < 1$.

Hence the given series is convergent.

CASE II. Here $u_n^{\frac{1}{n}} > 1$

$\Rightarrow \quad u_n > 1$

$\Rightarrow \quad u_1 + u_2 + u_3 + \ldots + u_n > n$

$$\Rightarrow \quad \lim_{n \to \infty} (u_1 + u_2 + u_3 + \ldots + u_n) > \lim_{n \to \infty} n = \infty.$$

Hence $\sum u_n$ diverges.

No definite conclusion can be drawn regarding the nature of $\sum_{n=1}^{\infty} u_n$ when $\lim_{n \to \infty} u_n^{\frac{1}{n}} = 1$.

WORKED OUT EXAMPLES

1. *Test the convergence of the series—*
(श्रेणी के अभिसरण की जांच करें)—

$$\frac{1}{4} + \frac{1.3}{4.7} + \frac{1.3.5}{4.7.10} + \frac{1.3.5.7}{4.7.10.13} + \ldots$$

(P U 1950; B U '60; M U '63, '68)

Solution. Here $u_n = \frac{1.3.5.7 \ldots (2n-1)}{4.7.10.13 \ldots (3n+1)}$.

So, we get

$$\frac{u_n}{u_{n+1}} = \frac{1.3.5.\ldots.(2n-1)}{4.7.10.\ldots.(3n+1)} \cdot \frac{4.7.10.\ldots.(3n+3+1)}{1.3.5.\ldots.(2n+2-1)}$$

$$= \frac{3n+4}{2n+1} = \frac{3+\dfrac{4}{n}}{2+\dfrac{1}{n}}.$$

$$\therefore \quad \underset{n\to\infty}{\mathrm{Lt}} \frac{u_n}{u_{n+1}} = \frac{3}{2} > 1. \quad \therefore \text{ The series is convergent.}$$

2. *Determine the convergency or divergency of the following series :*

$$1 + \frac{2^p}{\lfloor 2} + \frac{3^p}{\lfloor 3} + \frac{4^p}{\lfloor 4} + \ldots \text{ to } \infty, \text{ for all values of } p,$$

(B U 1954, '58, '63; P U '62, '68; B U '75; M U '71, '73)

Solution. Here the n^{th} term, $u_n = \dfrac{n^p}{\lfloor n}$.

$$\therefore \quad u_{n+1} = \frac{(n+1)^p}{\underline{|n+1}}.$$

$$\text{Now} \quad \frac{u_n}{u_{n+1}} = \frac{n^p}{\lfloor n} \cdot \frac{\underline{|n+1}}{(n+1)^p} = (n+1) . \frac{1}{\left(1+\dfrac{1}{n}\right)^p}.$$

$$\therefore \quad \underset{n\to\infty}{Lt} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left[(n+1) . \frac{1}{\left(1+\dfrac{1}{n}\right)^p} \right] = \infty,$$

which is greater than 1 for all p.

Hence, by D'Alembert's ratio test, the given series is convergent for all p.

3. (i) *Determine the convergence—*

$$2x + \frac{3x}{8} + \frac{4x^3}{27} + \ldots + \frac{(n+1)}{n^3} x^n + \ldots \text{ to } \infty,$$

$(x > 0)$.

(M U 1965, '73; B U '61, '78, '80;
P U '61; R U '69; Mith. U. '85)

Solution. Here the n^{th} term, $u_n = \dfrac{n+1}{n^3} x^n$.

$$\therefore \quad u_{n+1} = \frac{(n+1)+1}{(n+1)^3} . x^{n+1}.$$

Now $$\frac{u_n}{u_{n+1}} = \frac{n+1}{n+2}.\left(\frac{n+1}{n}\right)^3.\frac{1}{x}$$

$$= \frac{1+\frac{1}{n}}{1+\frac{2}{n}}.\left(1+\frac{1}{n}\right)^3.\frac{1}{x}.$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

If $x < 1$, the given series is convergent.
If $x > 1$, the given series is divergent.
If $x = 1$, the ratio test fails.

Now $u_n = \frac{n+1}{n^3}$, when $x = 1$.

Let $v_n = \frac{1}{n^2}$, the n^{th} term of the auxiliary series Σv_n.

Then $\frac{u_n}{v_n} = \frac{n+1}{n} = 1 + \frac{1}{n}$.

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1,$$

which is finite and non-zero.

So, by comparison test, Σu_n and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma \frac{1}{n^2}$, i.e. $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$ to ∞

is convergent as

$$\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots \text{to } \infty$$

is convergent when $p\ (= 2) > 1$.

$\therefore \quad \Sigma u_n$ is also convergent.

Hence the given series is convergent when $x \leqslant 1$ and divergent when $x > 1$.

3. (ii) *Discuss the convergence of the series—*

$$\frac{x}{1.2}+\frac{x^2}{3.4}+\frac{x^3}{5.6}+\frac{x^4}{7.8}+\ldots,\ (x>0).$$

(P U 1933; M U '70; B U '66,'72)

Solution. Here the n^{th} term, $u_n=\dfrac{x^n}{(2n-1).2n}$.

$$\therefore\ \frac{u_n}{u_{n+1}}=\frac{x^n}{(2n-1)2n}\Big/\frac{x^{n+1}}{(2n+1)\cdot(2n+2)}$$

$$=\frac{(2n+1)(2n+2)}{(2n-1)2n}\cdot\frac{1}{x}$$

$$=\frac{\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)}{\left(2-\frac{1}{n}\right)\cdot 2}\cdot\frac{1}{x}.$$

$$\therefore\ \operatorname*{Lt}_{n\to\infty}\frac{u_n}{u_{n+1}}=\frac{2.2}{2.2}\cdot\frac{1}{x}=\frac{1}{x}.$$

$\therefore$ If $x<1$, the series is convergent for then

$$\operatorname*{Lt}_{n\to\infty}\frac{u_n}{u_{n+1}}>1.$$

If $x>1$, the series is divergent for then $\operatorname*{Lt}_{n\to\infty}\dfrac{u_n}{u_{n+1}}<1$.

If $x=1$, $u_n=\dfrac{1}{(2n-1)2n}=\dfrac{1}{n^2}\cdot\dfrac{1}{\left(2-\frac{1}{n}\right)\cdot 2}$.

Put $\qquad v_n=\dfrac{1}{n^2}$.

$$\therefore\ \operatorname*{Lt}_{n\to\infty}\frac{u_n}{v_n}=\operatorname*{Lt}_{n\to\infty}\frac{1}{\left(2-\frac{1}{n}\right).2}=\frac{1}{4}.$$

$\therefore$ $\sum u_n$ and $\sum v_n$ converge or diverge together. But $\sum v_n$ is convergent, as $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots$ is convergent obtained by comparing with the series $1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$ which is convergent if $p > 1$ and divergent if $p \leqslant 1$.

$\therefore$ Σu_n is also convergent.

$\therefore$ If $x = 1$, the series is convergent.

3. **(iii)** *Determine the convergence of the series—*

$$1 + \frac{2}{5}x + \frac{6}{9}x^2 + \frac{14}{17}x^3 + \dots + \frac{2^n - 2}{2^n + 1}.x^{n-1} + \dots \textit{to } \infty,$$

$(x > 0)$

(P U 1948; Baroda Engg. '60)

Solution. Here the n^{th} term, $u_n = \frac{2^n - 2}{2^n + 1}x^{n-1}$.

$$\therefore \quad \frac{u_n}{u_{n+1}} = \left(\frac{2^n - 2}{2^n + 1}\right)x^{n-1} \Big/ \left(\frac{2^{n+1} - 2}{2^{n+1} + 1}\right)x^n$$

$$= \frac{2^n - 2}{2^{n+1} - 2} \cdot \frac{2^{n+1} + 1}{2n + 1} \cdot \frac{1}{x}$$

$$= \frac{1 - \frac{2}{2^n}}{2 - \frac{2}{2^n}} \cdot \frac{2 + \frac{1}{2^n}}{1 + \frac{1}{2^n}} \cdot \frac{1}{x}.$$

$$\therefore \quad \underset{n \to \infty}{\text{Lt}} \frac{u_n}{u_{n+1}} = \frac{1}{2} \cdot \frac{2}{1} \cdot \frac{1}{x} = \frac{1}{x}.$$

$\therefore$ If $x < 1$, the series is convergent.

If $x > 1$, the series is divergent.

If $x = 1$, D'Alembert's test fails. Then $u_n = \frac{2^n - 2}{2^n + 1}$.

$$\text{But } u_n = \frac{2^n - 2}{2n + 1} > \frac{2^n - 2^{n-1}}{2^n + 2^n} \text{ for } n \geqslant 3$$

$$= \frac{2^{n-1}}{2^{n+1}} = \frac{1}{2^2} = \frac{1}{4}.$$

$\therefore$ $u_n > \frac{1}{4}$ from and after the 3 term at least.

$\therefore$ $1 + \frac{2}{5} + \frac{6}{9} + \frac{14}{17} + \dots$

$$> 1 + \tfrac{2}{5} + \left(\tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} + \dots \text{to } \infty\right).$$

$\therefore$ The series is divergent.

$\therefore$ If $x = 1$, the series is divergent.

4. (i) *Test the series*

$$\sum_{n=1}^{\infty} \frac{x^n}{1+n^2} \text{ for all } x > 0.$$

(Bh U 1963; B U '62, '74, '77; P U '53, '70; Mith U '76)

Solution. Here the n^{th} term, $u_n = \dfrac{x^n}{1+n^2}$.

$$\therefore \quad u_{n+1} = \frac{x^{n+1}}{1+(n+1)^2}.$$

Now $$\frac{u_n}{u_{n+1}} = \frac{n^2+2n+2}{n^2+1} \cdot \frac{1}{x} = \frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}} \cdot \frac{1}{x}.$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}} \cdot \frac{1}{x} = \frac{1}{x}.$$

Now we apply D'Alembert's ratio test.

If $x < 1$, $\lim\limits_{n\to\infty} \dfrac{u_n}{u_{n+1}} > 1$ and so the series is convergent.

If $x > 1$, $\lim\limits_{n\to\infty} \dfrac{u_n}{u_{n+1}} < 1$ and so the series is divergent.

If $x = 1$, $\lim\limits_{n\to\infty} \dfrac{u_n}{u_{n+1}} = 1$ and so the ratio test fails.

Now, $u_n = \dfrac{1}{1+n^2}$ when $x = 1$.

Let $v_n = \dfrac{1}{n^2}$, the n^{th} term of the auxiliary series Σv_n.

Then $$\frac{u_n}{v_n} = \frac{n^2}{1+n^2} = \frac{1}{\frac{1}{n^2}+1}.$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\frac{1}{n^2}+1} = 1,$$

which is finite and non-zero.

So, by comparison test, Σu_n and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma \frac{1}{n^2}$ i.e. $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$ to ∞

is convergent as $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$ to ∞ is convergent when $p\ (= 2) > 1$.

Therefore, Σu_n is also convergent.

Hence the given series is convergent or divergent according as $x \leqslant 1$ or $x > 1$.

4. **(ii)** *Test the convergency of the series whose general term is* $\frac{n^2}{n^3+1} x^{n-1}$, $(x > 0)$.

(Bh U 1965; B U '58; M U '68; R U '73)

Solution. Here the n^{th} term, $u_n = \frac{n^2}{n^3+1} x^{n-1}$.

$$\therefore \quad u_{n+1} = \frac{(n+1)^2}{(n+1)^3+1} x^n.$$

$$\text{Now} \quad \frac{u_n}{u_{n+1}} = \frac{n^2}{(n+1)^2} \cdot \frac{(n+1)^3+1}{n^3+1} \cdot \frac{1}{x}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^2} \cdot \frac{\left(1+\frac{1}{n}\right)^3 + \frac{1}{n^3}}{1+\frac{1}{n^3}} \cdot \frac{1}{x}.$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

If $x < 1$, the given series is convergent.

If $x > 1$, the given series is divergent.

If $x = 1$, the ratio test fails.

Now $u_n = \dfrac{n^2}{n^3+1}$, when $x = 1$.

Let $v_n = \dfrac{1}{n}$, the n^{th} term of the auxiliary series Σv_n.

$$\text{Then } \frac{u_n}{v_n} = \frac{n^3}{n^3+1} = \frac{1}{1+\dfrac{1}{n^3}}.$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{1+\dfrac{1}{n^3}} = 1,$$

which is finite and non-zero.

So, by comparison test, Σu_n and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma \dfrac{1}{n}$, i.e. $\dfrac{1}{1} + \dfrac{1}{2} + \dfrac{1}{3} + \ldots$ to ∞

is divergent as $\dfrac{1}{1^p} + \dfrac{1}{2^p} + \dfrac{1}{3^p} + \ldots$ to ∞ is divergent when

$$p = 1.$$

$\therefore$ Σu_n is also divergent.

Hence the given series is convergent when $x < 1$ and divergent when $x \geqslant 1$.

4. (iii) *Test for convergence or divergence the series whose n^{th} term is* $\dfrac{\sqrt{n^2+1}}{\sqrt{n^4+1}}x^n$, $(x > 0)$. (P U 1960; Bh U '71)

Solution. Here $u_n = \dfrac{\sqrt{n^2+1}}{\sqrt{n^4+1}}x^n$; $\therefore u_{n+1} = \dfrac{\sqrt{(n+1)^2+1}}{\sqrt{(n+1)^4+1}}x^{n+1}$.

$$\therefore \frac{u_n}{u_{n+1}} = \frac{\sqrt{n^2+1}}{\sqrt{n^4+1}}x^n \Bigg/ \frac{\sqrt{(n+1)^2+1}}{\sqrt{(n+1)^4+1}}x^{n+1}.$$

$$\therefore \frac{u_n}{u_{n+1}} = \sqrt{\frac{n^2+1}{(n+1)^2+1}} \cdot \sqrt{\frac{(n+1)^4+1}{n^4+1}} \cdot \frac{1}{x}$$

$$=\sqrt{\frac{1+\frac{1}{n^2}}{\left(1+\frac{1}{n}\right)^2+\frac{1}{n^3}}}\cdot\sqrt{\frac{\left(1+\frac{1}{n}\right)^4+\frac{1}{n^4}}{1+\frac{1}{n^4}}}\cdot\frac{1}{x}.$$

$\therefore\ \operatorname*{Lt}_{n\to\infty}\frac{u_n}{u_{n+1}}=\frac{1}{x}$. $\therefore$ If $x<1$, the series is convergent and if $x>1$, the series is divergent.

If $x=1$, $u_n=\dfrac{\sqrt{n^2+1}}{\sqrt{n^4+1}}=\dfrac{1}{n}\cdot\sqrt{\dfrac{1+\frac{1}{n^2}}{1+\frac{1}{n^4}}}$. Put $v_n=\dfrac{1}{n}$.

Then $\operatorname*{Lt}_{n\to\infty}\frac{u_n}{v_n}=1$. So Σu_n and Σv_n converge or diverge together.

But $\Sigma v_n=\sum\frac{1}{n}$ is divergent. So Σu_n is also divergent.

$\therefore$ If $x=1$, the series is divergent.

5. (i) *Test for convergence the series*

$$\Sigma\,\frac{(1+x)(2+x)(3+x)\ldots.(n+x)}{(2+x)(4+x)(6+x)\ldots.(2n+x)}.\quad \text{(P U 1964 H)}$$

Solution. Let the given series be denoted by Σu_n. Clearly the given series has all the terms of the same sign for all possible values of x, after some stage.

Then $u_n=\dfrac{(1+x)(2+x)(3+x)\ldots.(n+x)}{(2+x)(4+x)(6+x)\ldots(2n+x)}$ and

$$u_{n+1}=\frac{(1+x)(2+x)(3+x)\ldots.(n+x)(n+1+x)}{(2+x)(4+x)(6+x)\ldots.(2n+x)(2n+2+x)}.$$

$$\therefore\ \frac{u_n}{u_{n+1}}=\frac{2n+2+x}{n+1+x}=\frac{2+\frac{2}{n}+\frac{x}{n}}{1+\frac{1}{n}+\frac{x}{n}}$$

$\Rightarrow\ \lim\limits_{n\to\infty}\dfrac{u_n}{u_{n+1}}=2$, which is greater than unity.

Hence, by D'Alembert's ratio test, the given series is convergent.

5. (ii) *Test the convergence of the series*

$$\sum_{n=2}^{\infty}\left\{\frac{1}{\sqrt{n^2-1}}-\frac{1}{\sqrt{n^2+1}}\right\}x^n \text{ for all values of } x.$$

(P U 1958)

Solution. Here

$$u_n=\left(\frac{1}{\sqrt{n^2-1}}-\frac{1}{\sqrt{n^2+1}}\right)x^n=\frac{\sqrt{n^2+1}-\sqrt{n^2-1}}{\sqrt{n^4-1}}x^n$$

$$=\frac{\sqrt{1+\frac{1}{n^2}}-\sqrt{1-\frac{1}{n^2}}}{n.\sqrt{1-\frac{1}{n^4}}}x^n$$

$$=\frac{\left(1+\frac{1}{n^2}\right)-\left(1-\frac{1}{n^2}\right)}{n.\sqrt{1-\frac{1}{n^4}}\left(\sqrt{1+\frac{1}{n^2}}+\sqrt{1-\frac{1}{n^2}}\right)}x^n$$

$$=\frac{2}{\sqrt{1-\frac{1}{n^4}}\left(\sqrt{1+\frac{1}{n^2}}+\sqrt{1-\frac{1}{n^2}}\right)}\cdot\frac{x^n}{n^3}.$$

Let $v_n=\frac{x^n}{n^3}$; then $\underset{n\to\infty}{\text{Lt}}\frac{u_n}{v_n}=1$.

$\therefore$ Σu_n and Σv_n converge or diverge together.

Now, $\frac{v_n}{v_{n+1}}=\left(\frac{n+1}{n}\right)^3\cdot\frac{1}{x}$; $\therefore$ $\underset{n\to\infty}{\text{Lt}}\frac{v_n}{v_{n+1}}=\frac{1}{x}$.

$\therefore$ If $x<1$, $\underset{n\to\infty}{\text{Lt}}\frac{v_n}{v_{n+1}}>1$ and Σv_n is convergent.

And if $x>1$, Σv_n is divergent.

If $x=1$, $v_n=\frac{1}{n^3}$ and $\therefore$ Σv_n is convergent.

$\therefore$ Σu_n is convergent if $x\leqslant 1$ and divergent if $x>1$.

6. *Discuss the convergency and divergency of the series whose n^{th} term is* $\dfrac{a^n}{x^n + a^n}$.

Solution. Here $u_n = \dfrac{a^n}{a^n + x^n}$

and $$u_{n+1} = \frac{a^{n+1}}{a^{n+1} + x^{n+1}}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{a^{n+1} + x^{n+1}}{a^n + x^n} \cdot \frac{1}{a}$$

$$= \frac{\left(\dfrac{a}{x}\right)^{n+1} + 1}{\left(\dfrac{a}{x}\right)^n + 1} \cdot \frac{x}{a}.$$

Let $x > a$.

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{x}{a}.$$

If $a < x$, the series is convergent.

Again $$\frac{u_n}{u_{n+1}} = \frac{1 + \left(\dfrac{x}{a}\right)^{n+1}}{1 + \left(\dfrac{x}{a}\right)^n}.$$

Let $a > x$.

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1 \text{ if } x < a.$$

The test fails in this case.

But $$u_n = \frac{1}{1 + \left(\dfrac{x}{a}\right)^n}.$$

$\therefore \quad \lim_{n\to\infty} u_n = 1$ which is not equal to zero.

Hence $\sum u_n$ is divergent.

When $x = a$, the given series becomes

$$\tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \ldots \text{to } \infty,$$

which is clearly divergent.

Hence the given series is convergent if $x > a$ and divergent if $x \leqslant a$.

7. *Discuss the convergence of the series for real values of x*—

$$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \ldots .\textit{to } \infty.$$

(P U 1978, '80; R U '68; M U '70; B U '66)

Solution. Clearly, for all real values of x, the series is of positive terms.

Here the n^{th} term, $u_n = \dfrac{x^{2n-2}}{(n+1)\sqrt{n}}$.

$$\therefore \quad u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}.$$

$$\text{Now} \quad \frac{u_n}{u_{n+1}} = \frac{(n+2)\sqrt{n+1}}{(n+1)\sqrt{n}} \cdot \frac{1}{x^2}$$

$$= \frac{\left(1+\dfrac{2}{n}\right)}{\left(1+\dfrac{1}{n}\right)} \cdot \sqrt{1+\frac{1}{n}} \cdot \frac{1}{x^2}.$$

$$\therefore \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}.$$

If $x^2 < 1$, i.e. $-1 < x < 1$, the series is convergent.

If $x^2 > 1$, i.e. $x > 1$ or $x < -1$, the series is divergent.

If $x^2 = 1$, i.e. $x = \pm 1$, the ratio test fails.

Now $u_n = \dfrac{1}{(n+1)\sqrt{n}}$, when $x = \pm 1$.

Let $v_n = \dfrac{1}{n^{\frac{3}{2}}}$, the n^{th} term of the auxiliary series $\sum v_n$.

Then $\dfrac{u_n}{v_n} = \dfrac{n}{n+1} = \dfrac{1}{1+\dfrac{1}{n}}$.

$\therefore \quad \lim\limits_{n \to \infty} \dfrac{u_n}{v_n} = 1$, which is finite and non-zero.

So, by comparison test, Σu_n and Σv_n converge or diverge together.

But $\Sigma v_n = \Sigma \frac{1}{n^{\frac{3}{2}}}$, i.e. $\frac{1}{1^{\frac{3}{2}}} + \frac{1}{2^{\frac{3}{2}}} + \frac{1}{3^{\frac{3}{2}}} + \ldots$ to ∞

is convergent as $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$ to ∞ is convergent when $p\,(= \frac{3}{2}) > 1$.

$\therefore$ Σu_n is also convergent.

Hence the given series is convergent when $-1 \leqslant x \leqslant 1$. and divergent when $x > 1$ or $x < -1$.

8. *Test the convergence of* $\sum_{1}^{\infty} \frac{\lfloor n}{n^n}$

(M U 1966 H; B U '62 H; Bh U '62 H)

Solution. Let the given series be denoted by Σu_n.

Then $u_n = \frac{\lfloor n}{n^n}$ and $u_{n+1} = \frac{\underline{|n+1}}{(n+1)^{n+1}}$.

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(n+1)^{n+1}}{n^n} \cdot \frac{\lfloor n}{\lfloor n+1}$$

$$= \frac{(n+1)^n(n+1)}{n^n} \cdot \frac{\lfloor n}{(n+1)\lfloor n} = \left(1 + \frac{1}{n}\right)^n.$$

$\Rightarrow$ $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = e$ which is greater than unity, as e lies between 2 and 3.

Hence, by D'Alembert's ratio test, the given series is convergent.

9. *Test for convergency and divergency the series*

$\frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \frac{4}{5}x^5 + \ldots$ *to infinity* $(x > 0)$,

and find its sum when it is convergent.

(P U 1957)

Solution. Here $u_n = \frac{n}{n+1} x^{n+1}$.

$$\therefore \quad \frac{u_n}{u_{n+1}} = \left(\frac{n}{n+1} x^{n+1}\right) \Big/ \left(\frac{n+1}{n+2} x^{n+2}\right) = \frac{n(n+2)}{(n+1)^2} \cdot \frac{1}{x}$$

$$= \frac{1 + \dfrac{2}{n}}{\left(1 + \dfrac{1}{n}\right)^2} \cdot \frac{1}{x}.$$

$$\therefore \quad \operatorname*{Lt}_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

$\therefore$ If $\dfrac{1}{x} > 1$ i.e. $x < 1$, the series is convergent.

If $x > 1$, the series is divergent.

If $x = 1$, D'Alembert's test fails.

Then $u_n = \dfrac{n}{n+1} > \dfrac{n}{2n} = \dfrac{1}{2}$ for all $n \geqslant 2$.

$\therefore$ The series $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \ldots . > \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$ The series on the right hand side is divergent. Hence $\sum u_n$ is divergent.

$\therefore$ It is convergent only when $x < 1$.

Then $\frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{4}x^4 + \frac{4}{5}x^5 + \ldots .$

$$= (1 - \tfrac{1}{2})x^2 + (1 - \tfrac{1}{3})x^3 + (1 - \tfrac{1}{4})x^4 + (1 - \tfrac{1}{5})x^5 + \ldots .$$

$$= (x^2 + x^3 + x^4 + \ldots) + (-\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \ldots)$$

$$= (x + x^2 + x^3 + x^4 + \ldots) + (-x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \ldots .)$$

$$= \frac{x}{1-x} + \log(1-x), \text{ as } x < 1.$$

10. (i) *Test for convergence of the series whose general term is*

$$\frac{n^{n^2}}{(n+1)^{n^2}}.$$

(Allahabad 1960; Delhi '55 H; B U '67 H)

Solution. Here the n^{th} term, $u_n = \dfrac{1}{\left(1 + \dfrac{1}{n}\right)^{n^2}}$.

$$\therefore \quad u_n^{\frac{1}{n}} = \frac{1}{\left(1 + \dfrac{1}{n}\right)^n}.$$

$$\therefore \lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

i.e. $\lim_{n\to\infty} \sqrt[n]{u_n} = \frac{1}{e} < 1$, as $2 < e < 3$.

Hence by Cauchy's n^{th} root test, the given series is convergent.

10. **(ii)** *Test the series for convergence*

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\ldots\text{to } \infty.$$

(B U 1957 H; Allahabad '41; Raj '55; Nagpur '57]

Solution. Here the n^{th} term,

$$u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-n}$$

$$= \left[\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n}\right)\right]^{-n}$$

$$\therefore \quad u_n^{\frac{1}{n}} = \left[\left(1+\frac{1}{n}\right)^{n+1} - \left(1+\frac{1}{n}\right)\right]^{-1}$$

$$\therefore \quad \lim_{n\to\infty} u_n^{\frac{1}{n}} = (e-1)^{-1}, \text{ as } \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = e.$$

i.e. $\lim_{n\to\infty} \sqrt[n]{u_n} = \frac{1}{e-1} < 1$, as $3 > e > 2$.

Hence, by Cauchy's n^{th} root test, the given series is convergent.

Problem-Set 4

Discuss the convergency and divergency of the following series—

1. $1+\frac{2}{3^2}+\frac{2.4}{3^2.5^2}+\frac{2.4.6}{3^2.5^2.7^2}+\ldots$ to ∞.

2. (i) $1+\frac{2^{-p}}{\lfloor 2}+\frac{3^{-p}}{\lfloor 3}+\ldots+\frac{n^{-p}}{\lfloor n}+\ldots$ to ∞.

(M U 1972)

(ii) $\frac{\lfloor 1}{1^1}+\frac{\lfloor 2}{2^2}+\frac{\lfloor 3}{3^3}+\ldots$ (M U 1971)

3. (i) $\frac{3}{4}+\frac{1}{2}x+\frac{1}{12}x^2+\ldots+\frac{2n-5}{n^3-5n}x^{n-1}+\ldots$ to ∞,

$(x>0)$

(P U 1954, '65)

(ii) $1+\frac{x}{3}+\frac{x^2}{5}+\ldots$ to ∞, $(x>0)$.

(B U 1964, '69; P U '44)

(iii) $1+\frac{2^2-1}{2^2+1}x+\frac{3^2-2^2}{3^2+2^2}x^2+\ldots+\frac{n^2-(n-1)^2}{n^2+(n-1)^2}x^{n-1}$

$+\ldots$ to ∞, $(x>0)$. (P U 1952, '56)

(iv) $1+3x+5x^2+7x^3+9x^4+\ldots$ to ∞, $(x>0)$.

(M U 1964; Roorkee Engg. '70)

(v) $\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+\frac{x^4}{4.5}+\ldots$ to ∞, $(x>0)$.

(R U 1965, '72, '74; P U '57; M U '72, '75;
Gorakhpur Engg. '72)

(vi) $\frac{x}{2}+\frac{2x^2}{5}+\frac{3x^3}{10}+\ldots+\frac{nx^n}{n^2+1}+\ldots$, $(x>0)$

(M U 1972)

(vii) $\frac{2}{1}x+\frac{7}{15}x^2+\frac{12}{53}x^3+\ldots+\frac{5n-3}{2n^3-1}x^n+\ldots$ to ∞,

$(x>0)$.

(Bh U 1969)

(viii) $\frac{2}{2}+\frac{4}{5}x+\frac{6}{10}x^2+\frac{8}{17}x^3+\frac{10}{26}x^4+\ldots$ to ∞, $(x>0)$.

(R U 1969)

(ix) $\frac{1}{2}+\frac{x}{3\sqrt{2}}+\frac{x^2}{4\sqrt{3}}+\frac{x^3}{5\sqrt{4}}+\ldots$ to ∞, $(x>0)$

(B U 1966)

(x) $1 + \frac{3}{2}x + \frac{5}{9}x^2 + \frac{7}{28}x^3 + \frac{9}{65}x^4 + \ldots\text{to } \infty, \quad (x > 0).$

(B U 1965; P U '78; '85)

(xi) $1 + \frac{1}{2}x + \frac{1}{5}x^2 + \frac{1}{10}x^3 + \ldots + \frac{x^n}{n^2+1} + \ldots\text{to } \infty,$

$(x > 0).$

(Roorkee Engg. 1971)

4. Discuss the convergence of the series whose general term (or n^{th} term) is the following $(x > 0)$—

(i) $\frac{x^n}{\sqrt{n^2+1}}$. (B U 1960) (ii) $\frac{x^n}{(2n+1)^p}$.

(B U 1961; R U '65)

(iii) $\frac{x^n}{2n^2+3}$. (B U 1969) (iv) $\frac{n+1}{n^2+1}x^n$. (P U 1953)

(v) $\frac{x^n}{n+\sqrt{1+n^2}}$. (P U 1963)

(vi) $\frac{\sqrt{n+1}}{\sqrt{n^3+1}}x^n$. (P U 1956)

5. Discuss the convergence of the following series—

(i) $\sum_1^\infty \frac{n^2-1}{n^2+1}x^n, \quad (x > 0).$ (B U 1957; P U '51)

(ii) $\sum_{n=1}^\infty \frac{1}{1+nx^n}$ for all real values of x. (P U 1958)

6. Discuss the convergence for $x \in R$ if $a > 0$—

$$\frac{1}{x \quad 1} + \frac{a}{x+a} + \frac{a^2}{x+a^2} + \frac{a^3}{x+a^3} + \ldots\text{to } \infty.$$

(M U 1964, '66)

7. Test the convergence of the series for all real values of x—

$$\frac{1}{1.3} + \frac{x^2}{3.5} + \frac{x^4}{5.7} + \frac{x^6}{7.9} + \ldots\text{to } \infty.$$

8. Test the convergence of $\sum_{1}^{\infty} \dfrac{n^n}{n!}$.

9. Examine the convergence of the series—

(i) $\sum_{1}^{\infty} \left(1+\dfrac{1}{n}\right)^{-n^2}$ (ii) $\sum_{2}^{\infty} \dfrac{1}{(\log n)^n}$.

3.11. More Tests for Convergence of Series of Positive Terms

Theorem 3.15. *Second Comparison Test.* (M U 1963 H)

Statement. If $\sum u_n$ and $\sum v_n$ be two series of positive terms, then $\sum u_n$ will be convergent when $\sum v_n$ is convergent if after some particular term $\dfrac{u_n}{u_{n-1}} < \dfrac{v_n}{v_{n-1}}$.

Also $\sum u_n$ will be divergent when $\sum v_n$ is divergent if after some particular term $\dfrac{u_n}{u_{n-1}} > \dfrac{v_n}{v_{n-1}}$.

Proof. Let the particular terms be u_1 and v_1.

CASE I. Let $\dfrac{u_2}{u_1} < \dfrac{v_2}{v_1}$, $\dfrac{u_3}{u_2} < \dfrac{v_3}{v_2}, \ldots$.

$$\begin{aligned}\text{Then } u_1+u_2+u_3+\ldots &= u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_1}+\ldots\right)\\ &= u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}.\frac{u_2}{u_1}+\ldots\right)\\ &< u_1\left(1+\frac{v_2}{v_1}+\frac{v_3}{v_2}.\frac{v_2}{v_1}+\ldots\right);\end{aligned}$$

that is, $u_1+u_2+u_3+\ldots < \dfrac{u_1}{v_1}(v_1+v_2+v_3+\ldots)$.

From this we can say that if $\sum v_n$ is convergent, then $\sum u_n$ is also convergent.

CASE II. Let $\dfrac{u_2}{u_1} > \dfrac{v_2}{v_1}$, $\dfrac{u_3}{u_2} > \dfrac{v_3}{v_2}, \ldots$.

$$\text{Then } u_1 + u_2 + u_3 + \ldots = u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \ldots\right)$$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}.\frac{u_2}{u_1} + \ldots\right)$$

$$> u_1\left(1 + \frac{v_2}{v_1} + \frac{v_3}{v_2}.\frac{v_2}{v_1} + \ldots\right)$$

that is, $u_1 + u_2 + u_3 + \ldots > \frac{u_1}{v_1}(v_1 + v_2 + v_3 + \ldots)$.

This shows that if $\sum v_n$ is divergent, then $\sum u_n$ is also divergent.

Note. This theorem can be enunciated in the limit form as follows :

The u-series will be convergent when the v-series is convergent provided that

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} > \lim_{n\to\infty} \frac{v_n}{v_{n+1}};$$

and the u-series will be divergent when the v-series is divergent provided that

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} < \lim_{n\to\infty} \frac{v_n}{v_{n+1}}.$$

Theorem. 3.16. *Raabe's Test.* (M U 1963 H, '70 H; R U '69 H, '74 H; Bh. U '64 H, '68 H; P U 51 H; B U '53 H)

Statement. The series $\sum u_n$ of positive terms is convergent if

$$\lim_{n\to\infty} \left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} > 1,$$

$$\text{or divergent if } \lim_{n\to\infty} \left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} < 1.$$

Proof. The given series is

$$u_1 + u_2 + u_3 + \ldots + u_n + u_{n+1} + \ldots \text{to } \infty.$$

Let the auxiliary series $\sum v_n$ be $\sum \frac{1}{n^p}$

$$\therefore \quad v_n = \frac{1}{n^p} \text{ and } v_{n+1} = \frac{1}{(n+1)^p}.$$

We know that $\Sigma\, v_n$ is convergent or divergent according as $p >$ or < 1.

Hence, by the second comparison test, the given series Σu_n is convergent or divergent according as

$$\frac{u_n}{u_{n+1}} > \text{ or } < \frac{v_n}{v_{n+1}}$$

$$\Rightarrow \frac{u_n}{u_{u+1}} > \text{ or } < \left(\frac{n+1}{n}\right)^p$$

$$\Rightarrow \frac{u_n}{u_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right)^p$$

$$\Rightarrow \frac{u_n}{u_{n+1}} > \text{ or } < 1 + p.\frac{1}{n} + \frac{p(p-1)}{2!}.\frac{1}{n^2} + \ldots \text{ to } \infty,$$

by Binomial Theorem

$$\Rightarrow \frac{u_n}{u_{n+1}} - 1 > \text{ or } < \frac{1}{n}\left[p + \frac{p(p-1)}{2!}.\frac{1}{n} + \ldots \text{to } \infty\right]$$

$$\Rightarrow n\left(\frac{u_n}{u_{n+1}} - 1\right) > \text{ or } < p + \frac{p(p-1)}{2!}.\frac{1}{n} + \ldots \text{to } \infty$$

$$\Rightarrow \lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} > \text{ or } < p$$

$$\Rightarrow \lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} > \text{ or } < 1.$$

Hence the theorem.

Note. (i) This test fails when the above limit is 1

(ii) Apply Raabe's test when D'Alembert's ratio test fails.

Theorem 3.17. *Logarithmic Ratio Test or Logarithmic Test.*

(P U 1954 H, '57 H, '62 H; R U '64 H)

Statement. The series Σu_n of positive terms is convergent if

$$\lim_{n\to\infty}\left(n \log \frac{u_n}{u_{n+1}}\right) > 1,$$

or divergent if

$$\lim_{n\to\infty}\left\{n\log\frac{u_n}{u_{n+1}}\right\} < 1.$$

Proof. The given series is

$$u_1 + u_2 + u_3 + \ldots + u_n + u_{n+1} + \ldots \text{to } \infty.$$

Let the auxiliary series $\sum v_n$ be $\sum \frac{1}{n^p}$.

$$\therefore\quad v_n = \frac{1}{n^p} \text{ and } v_{n+1} = \frac{1}{(n+1)^p}.$$

We know that $\sum v_n$ is convergent or divergent according as $p >$ or < 1.

Hence, by the second comparison test, the given series $\sum u_n$ is convergent or divergent according as

$$\frac{u_n}{u_{n+1}} \quad > \text{ or } < \quad \frac{v_n}{v_{n+1}}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} \quad > \text{ or } < \quad \left(\frac{n+1}{n}\right)^p$$

$$\Rightarrow \quad \log\frac{u_n}{u_{n+1}} \quad > \text{ or } < \quad p\log\left(1+\frac{1}{n}\right)$$

$$\Rightarrow \quad \log\frac{u_n}{u_{n+1}} \quad > \text{ or } < \quad p\left(\frac{1}{n} - \frac{1}{2n^2} + \ldots \text{to } \infty\right)$$

$$\Rightarrow \quad n\log\frac{u_n}{u_{n+1}} \quad > \text{ or } < \quad p\left(1 - \frac{1}{2n} + \ldots \text{to } \infty\right)$$

$$\Rightarrow \quad \lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}\right) \quad > \text{ or } < \quad p$$

$$\Rightarrow \quad \lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}\right) \quad > \text{ or } < \quad 1.$$

Hence the proposition.

Note. After the failure of D'Alembert's ratio test, the Logarithmic test should be applied to the series when n occurs as an exponent in $\frac{u_n}{u_{n+1}}$.

Theorem 3.18. *Cauchy's Condensation Test.*

(M U 1968 H, '70 H, '75 H, '80 H: '85;
Bh U '63 H, '65 H, '67 H, '69 H, '72 H;
B U '55 H, '59 H, '61 H, '68 H;
P U '51 H, '54 H, '58 H, '61 H, '64 H, '68 H, '70 H, '73 H;
B P S C '83, '80)

Statement. If the function $f(n)$ be positive for all positive integral values of n, and continually decreases as n increases and if a be a positive integer greater than unity, then the two infinite series $\sum f(n)$ and $\sum a^n f(a^n)$, that is,

$$f(1) + f(2) + f(3) + \ldots + f(n) + \ldots \text{to } \infty$$

and $af(a) + a^2 f(a^2) + a^3 f(a^3) + \ldots + a^n f(a^n) + \ldots$ to ∞

are both convergent or both divergent.

Proof. As $\sum f(n)$ is a series of positive terms and the convergence of a series of positive terms is unaffected by grouping the terms, let the terms of $\sum f(n)$ be grouped as follows:

$$\begin{aligned} &\{f(1) + f(2) + f(3) + \ldots + f(a)\} \\ &\quad + \{f(a+1) + f(a+2) + f(a+3) + \ldots + f(a^2)\} \\ &\quad + \{f(a^2+1) + f(a^2+2) + f(a^2+3) + \ldots + f(a^3)\} \\ &\quad + \ldots \\ &\quad + \{f(a^{n-1}+1) + f(a^{n-1}+2) + f(a^{n-1}+3) + \ldots + f(a^n)\} \\ &\quad + \ldots \end{aligned}$$

The number of terms in the nth group i.e.

$$\{f(a^{n-1}+1) + f(a^{n-1}+2) + f(a^{n-1}+3) + \ldots + f(a^n)\}$$

is $(a^n - a^{n-1})$, that is, $(a-1)a^{n-1}$.

Since $f(n)$ is a decreasing function, therefore, each term of the n^{th} group is less than $f(a^{n-1})$.

$$\therefore\ f(a^{n-1}+1) + f(a^{n-1}+2) + f(a^{n-1}+3) + \ldots + f(a^n) < a^{n-1}(a-1)f(a^{n-1}).$$

Putting $n = 1, 2, 3, \ldots$ successively, we get

$$f(2) + f(3) + f(4) + \ldots + f(a) < (a-1)f(1),$$

$$f(a+1) + f(a+2) + f(a+3) + \ldots + f(a^2) < (a-1)\,af(a)$$

.. ..

.. ..

Adding, we get

$$\Sigma f(n) - f(1) < (a-1) \Sigma a^n f(a^n) + (a-1) f(1).$$

$$\therefore \quad \Sigma f(n) < (a-1) \Sigma a^n f(a^n) + a f(1).$$

If $\Sigma a^n f(a^n)$ is convergent, then $(a-1) \Sigma a^n f(a^n)$ is a convergent series of positive terms as $a > 1$ and as $af(1)$ is a finite number, $\Sigma f(n)$ is also convergent.

Again each term of the n^{th} group is greater than the smallest term $f(a^n)$ of this group.

$$\therefore \quad f(a^{n-1}+1) + f(a^{n-1}+2) + f(a^{n-1}+3) + \ldots + f(a^n)$$
$$> (a-1)a^{n-1}f(a^n) \text{ i.e. } \frac{a-1}{a} a^n f(a^n).$$

Putting $n = 1, 2. 3, \ldots$ successively, we get

$$f(2) + f(3) + f(4) + \ldots + f(a) > \frac{a-1}{a} af(a),$$

$$f(a+1) + f(a+2) + f(a+3) + \ldots + f(a^2)$$
$$> \frac{a-1}{a} a^2 f(a^2),$$

.. ..

.. ..

Adding, we get

$$\Sigma f(n) - f(1) > \frac{a-1}{a} \Sigma a^n f(a^n).$$

If $\Sigma a^n f(a^n)$ is divergent then $\frac{a-1}{a} \Sigma a^n f(a^n)$ is also a divergent series of positive terms ($\because a > 1$) and consequently $\Sigma f(n)$ is also divergent.

Theorem 3.19. (*An Auxiliary Series for Convergence*)—

Statement. Prove that the series

$$\frac{1}{2(\log 2)^p} + \frac{1}{3(\log 3)^p} + \ldots + \frac{1}{n(\log n)^p} + \ldots + \text{ to } \infty$$

i.e. $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

Solution. Let the n^{th} term of the given series be denoted by $f(n)$.

Then $f(n) = \dfrac{1}{n(\log n)^p}$. Clearly $n(\log n)^p$, $p > 0$ is positive and increasing.

Also if $p < 0$, $\dfrac{n}{(\log n)^{-p}}$ is positive and increasing for n increases at a greater rate after some stage than $(\log n)^k$, $k > 0$. Thus $f(n)$ is positive and decreasing for all p.

Let a be any positive integer greater than unity.

$$\text{Then} \quad a^n f(a^n) = a^n . \frac{1}{a^n (\log a^n)^p}$$

$$= \frac{1}{(n \log a)^p} = \frac{1}{n^p (\log a)^p}$$

$$\therefore \quad \sum_{n=1}^{\infty} a^n f(a^n) = \sum_{n=1}^{\infty} \frac{1}{n^p (\log a)^p} = \frac{1}{(\log a)^p} \sum_{n=1}^{\infty} \frac{1}{n^p}.$$

But $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^p}$ is convergent or divergent according as

$$p > 1 \quad \text{or} \quad p \leqslant 1.$$

Therefore $\sum\limits_{n=1}^{\infty} a^n f(a^n)$ is convergent or divergent according as $p > 1$ or $p \leqslant 1$.

Hence, from Cauchy's Condensation test, the given series

$$\Sigma f(n), \text{ that is, } \Sigma \frac{1}{n(\log n)^p}$$

is convergent if $p > 1$ and divergent if $p \leqslant 1$.

Theorem 3.20. *De Morgan's and Bertrand's Test.*

(Bh U 1964 H; R U '67 H;
P U '50 H, '53 H, '62 H; B P S C '81)

Statement. The series, whose general term is u_n, is convergent or divergent according as

$$\lim_{n \to \infty} \left[\left\{ n\left(\frac{u_n}{u_{n+1}} - 1 \right) - 1 \right\} \log n \right] > 1, \text{ or } < 1.$$

Proof. The given series is

$$u_1 + u_2 + u_3 + \ldots + u_n + u_{n+1} + \ldots \text{to } \infty.$$

Let us compare the given series with the auxiliary series $\sum v_n$, which is

$$1 + \frac{1}{2(\log 2)^p} + \frac{1}{3(\log 3)^p} + \ldots + \frac{1}{n(\log n)^p} + \ldots \text{to } \infty.$$

We know that this series is convergent or divergent according as $p > 1$, or < 1.

Here $v_n = \dfrac{1}{n(\log n)^p}$ and $v_{n+1} = \dfrac{1}{(n+1)\{\log(n+1)\}^p}$.

Hence, by the second comparison test, the given series $\sum u_n$ is convergent or divergent according as

$$\frac{u_n}{u_{n+1}} > \text{ or } < \frac{v_n}{v_{n+1}}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} > \text{ or } < \left(\frac{n+1}{n}\right)\left\{\frac{\log(n+1)}{\log n}\right\}^p$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} > \text{ or } < \left(1 + \frac{1}{n}\right)\left\{\frac{\log\left[n\left(1 + \frac{1}{n}\right)\right]}{\log n}\right\}^p$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} > \text{ or } < \left(1 + \frac{1}{n}\right)\left\{\frac{\log n + \log\left(1 + \frac{1}{n}\right)}{\log n}\right\}^p$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} > \text{ or } < \left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{n \log n} - \frac{1}{2n^2 \log n} + \ldots\right)^p$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} > \text{ or } < \left(1 + \frac{1}{n}\right)\left(1 + \frac{p}{n \log n} - \frac{p}{2n^2 \log n} + \ldots\right),$$

by Binomial theorem

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} > \text{ or } < 1 + \frac{1}{n} + \frac{p}{n \log n} + \frac{p}{2n^2 \log n} + \ldots.$$

$$\Rightarrow \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) > \text{ or } < 1 + \frac{p}{n \log n} + \frac{p}{2n \log n} + \ldots.$$

$$\Rightarrow \quad \left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n > \text{ or } < p + \frac{p}{2n} + \ldots.$$

$$\Rightarrow \quad \lim_{n\to\infty} \left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n\right] > \text{ or } < p$$

$$\Rightarrow \quad \lim_{n\to\infty} \left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n\right] > \text{ or } < 1.$$

Hence the theorem.

Theorem 3.21. *Higher Logarithmic Test.*

Statement. The series $\sum u_n$ of positive terms is convergent or divergent according as

$$\lim_{n\to\infty} \left[\left\{n \log \frac{u_n}{u_{n+1}} - 1\right\} \log n\right] > 1 \text{ or } < 1.$$

Proof. The given series is

$$u_1 + u_2 + u_3 + \ldots + u_n + u_{n+1} + \ldots \text{to } \infty.$$

Let us compare this series with the auxiliary series $\sum v_n$, which is,

$$1 + \frac{1}{2(\log 2)^p} + \frac{1}{3(\log 3)^p} + \ldots + \frac{1}{n(\log n)^p} + \ldots \text{to } \infty.$$

Here $v_n = \dfrac{1}{n(\log n)^p}$, and $v_{n+1} = \dfrac{1}{(n+1)\{\log(n+1)\}^p}$.

We know that $\sum v_n$ is convergent or divergent according as

$$p > 1 \text{ or } < 1.$$

Hence, by the second comparison test, the given series $\sum u_n$ is convergent or divergent according as

$$\frac{u_n}{u_{n+1}} > \text{ or } < \frac{v_n}{v_{n+1}}$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} > \text{ or } < \frac{n+1}{n}\left\{\frac{\log(n+1)}{\log n}\right\}^p$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right)\left\{\frac{\log n + \log\left(1+\frac{1}{n}\right)}{\log n}\right\}^p$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right)\left(1+\frac{1}{n\log n} - \ldots\right)^p$$

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} > \text{ or } < \left(1+\frac{1}{n}\right)\left(1+\frac{p}{n\log n} - \ldots\right),$$

by Binomial theorem

$$\Rightarrow \quad \frac{u_n}{u_{n+1}} > \text{ or } < 1+\frac{1}{n}+\frac{p}{n\log n}+\frac{p}{n^2\log n} - \ldots.$$

$$\Rightarrow \quad \log\frac{u_n}{u_{n+1}} > \text{ or } < \log\left[1+\left(\frac{1}{n}+\frac{p}{n\log n}+\frac{p}{n^2\log n} - \ldots\right)\right]$$

$$\Rightarrow \quad \log\frac{u_n}{u_{n+1}} > \text{ or } < \frac{1}{n}+\frac{p}{n\log n}+\frac{p}{n^2\log n} - \ldots.$$

$$\Rightarrow \quad n\log\frac{u_n}{u_{n+1}} > \text{ or } < 1+\frac{p}{\log n}+\frac{p}{n\log n} - \ldots.$$

$$\Rightarrow \quad n\log\frac{u_n}{u_{n+1}} - 1 > \text{ or } < \frac{p}{\log n}+\frac{p}{n\log n} - \ldots.$$

$$\Rightarrow \quad \left(n\log\frac{u_n}{u_{n+1}} - 1\right)\log n > \text{ or } < p+\frac{p}{n} - \ldots.$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[\left\{n\log\frac{u_n}{u_{n+1}} - 1\right\}\log n\right] > \text{ or } < p$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[\left\{n\log\frac{u_n}{u_{n+1}} - 1\right\}\log n\right] > \text{ or } < 1.$$

Hence the theorem.

Theorem 3.22. *Kummer's Test.*

(M U 1965 H, '67 H, '69 H, '72 H, '74 H; R U '69 H, '73 H;
B U '72 H; P U '67 H, '70 H, '72 H)

Statement. Let $\sum u_n$ and $\sum \frac{1}{d_n}$ be two series of positive terms and let $\sum \frac{1}{d_n}$ be divergent; also suppose that

$$v_n = d_n \frac{u_n}{u_{n+1}} - d_{n+1}; \text{ then,}$$

(i) if a fixed number k can be found so that after a certain stage, say for $n \geqslant m$, $v_n > k > 0$, the series $\sum u_n$ is convergent;

(ii) if $v_n < 0$, for $n \geqslant m$, the series $\sum u_n$ is divergent.

Proof. (i) We have $v_n > k > 0$, for $n \geqslant m$

$$\Rightarrow \quad d_n \frac{u_n}{u_{n+1}} - d_{n+1} > k, \text{ for } n \geqslant m \text{ and } k > 0$$

$$\Rightarrow \quad d_n u_n - d_{n+1} u_{n+1} > k u_{n+1}, \text{ for } n \geqslant m \text{ and } k > 0.$$

Putting $m, m+1, m+2, \ldots, n-1$ in succession for n in the above inequality, we get

$$\begin{aligned}
d_m u_m - \cancel{d_{m+1} u_{m+1}} &> k u_{m+1}, \\
\cancel{d_{m+1} u_{m+1}} - \cancel{d_{m+2} u_{m+2}} &> k u_{m+2}, \\
\cancel{d_{m+2} u_{m+2}} - \cancel{d_{m+3} u_{m+3}} &> k u_{m+3}, \\
\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots \\
\cancel{d_{n-1} u_{n-1}} - d_n u_n &> k u_n.
\end{aligned}$$

Adding, we get $d_m u_m - d_n u_n > k(u_{m+1} + u_{m+2} + u_{m+3} + \ldots + u_n)$

$$\Rightarrow \quad u_{m+1} + u_{m+2} + u_{m+3} + \ldots + u_n < \frac{1}{k}(d_m u_m - d_n u_n) < \frac{1}{k} d_m u_m$$

$$\Rightarrow \quad (u_1 + u_2 + u_3 + \ldots + u_m + u_{m+1} + \ldots + u_n) - (u_1 + u_2 + u_3 + \ldots + u_m) < \frac{1}{k} d_m u_m.$$

If S_n be the sum to n terms of $\sum u_n$, then

$$S_n - S_m < \frac{1}{k} d_m u_m$$

$\Rightarrow \quad S_n < S_m + \frac{1}{k} d_m u_m$, which is a fixed number.

Hence $\sum u_n$ is convergent.

(ii) We have $v_n < 0$, for $n \geqslant m$

$$d_n \frac{u_n}{u_{n+1}} - d_{n+1} < 0, \text{ for } n \geqslant m$$

$$\Rightarrow \quad d_n u_n - d_{n+1} u_{n+1} < 0, \text{ for } n \geqslant m$$

$$\Rightarrow \quad d_n u_n < d_{n+1} u_{n+1}, \text{ for } n \geqslant m.$$

Putting $m, m+1, m+2, \ldots, n-1$ in succession for n in this inequality, we get

$$d_m u_m < d_{m+1} u_{m+1} < \ldots < d_n u_n$$

$$\therefore \quad d_n u_n > d_m u_m; \quad \therefore \quad u_n > d_m u_m \frac{1}{d_n}$$

$$\therefore \quad \sum_{n=1}^{\infty} u_n > d_m u_m \sum_{n=1}^{\infty} \frac{1}{d_n}.$$

Now $d_m u_m$ is a fixed number and $\sum_{n=1}^{\infty} \frac{1}{d_n}$ is divergent, therefore $\sum_{n=1}^{\infty} u_n$ is divergent.

Note. (a) If $v_n \to 0$ we cannot find k so that $v_n > k > 0$ for $n \geqslant m$, and the test fails.

In part (i) of the test, $\sum 1/d_n$ need not diverge; in fact, d_n may be any positive function of n.

(b) Limit form of Kummer's test

If $\sum 1/d_n$ is a divergent series of positive terms and $\sum u_n$ is a series of positive terms (> 0) and $\lim_{n \to \infty} \left(d_n . \frac{u_n}{u_{n+1}} - d_{n+1} \right) = l$ then $\sum u_n$ is convergent or divergent according as

$$l > 0 \text{ or } l < 0.$$

3.12. Deductions From Kummer's Test

The following tests can be deduced from Kummer's Test.

(i) *D' Alembert's Ratio Test.* (P U 1970 H)

Put $d_n = 1$ in Kummer's test.

Clearly Kummer's test is applicable as $\sum d_n$ is divergent.

$$\therefore \quad v_n = \frac{u_n}{u_{n+1}} - 1$$

Hence Σu_n is convergent if $\lim\limits_{n\to\infty} \dfrac{u_n}{u_{n+1}} > 1$

and Σu_n is divergent if $\lim\limits_{n\to\infty} \dfrac{u_n}{u_{n+1}} < 1$.

(ii) *Raabe's Test.* (P U 1970 H)

Put $d_n = n$ in Kummer's test.

In this case Kummer's test yields as $\Sigma \dfrac{1}{d_n}$ is divergent.

$$\therefore \quad d_{n+1} = n + 1.$$

$$\therefore \quad v_n = n\frac{u_n}{u_{u+1}} - (n+1) = n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1.$$

Hence Σu_n is convergent if $\lim\limits_{n\to\infty} \left\{n\left(\dfrac{u_n}{u_{n+1}} - 1\right)\right\} > 1$

and Σu_n is divergent if $\lim\limits_{n\to\infty} \left\{n\left(\dfrac{u_n}{u_{n+1}} - 1\right)\right\} < 1$.

(iii) *De Morgan and Bertrand's Test.*

Put $d_n = n \log n$ in Kummer's test.

In this case Kummer's test yields as $\Sigma \dfrac{1}{d_n}$ is divergent.

$$\begin{aligned}\therefore \quad d_{n+1} &= (n+1) \log (n+1) \\ &= (n+1) \log \left\{n\left(1 + \frac{1}{n}\right)\right\} \\ &= (n+1)\left[\log n + \log\left(1 + \frac{1}{n}\right)\right] \\ &= n \log n + \log n + (n+1)\log\left(1 + \frac{1}{n}\right).\end{aligned}$$

$$\therefore \quad v_n = n \log n . \frac{u_n}{u_{n+1}} - n \log n - \log n$$
$$- (n+1)\log\left(1 + \frac{1}{n}\right)$$

$$= n \log n\left(\frac{u_n}{u_{n+1}} - 1\right) - \log n - (n+1)\log\left(1 + \frac{1}{n}\right)$$

$$= \left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\}\log n - (n+1)\log\left(1 + \frac{1}{n}\right)$$

$$= \left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\}\log n - n\left\{\frac{1}{n} - \frac{1}{2n^2} + \ldots\right\} - \left\{\frac{1}{n} - \frac{1}{2n^2} + \ldots\right\}$$

Hence $\sum u_n$ is convergent or divergent according as

$$\lim_{n\to\infty}\left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\}\log n\right] > \text{or} < 1.$$

Theorem 3. 23. *Gauss's (Ratio) Test.*

(M U 1972 H; R U '68 H; Bh U '62 H; B U '85;
P U '66 H; '71 H; '73 H)

Statement. If $\sum u_n$ be a series of positive terms and if $\frac{u_n}{u_{n+1}}$ can be expressed in the form

$$\frac{u_n}{u_{n+1}} = 1 + \frac{a}{n} + \frac{b_n}{n^p},$$

where $p > 1$ and $|b_n| <$ a fixed number k or (in particular b_n tends to a finite limit as $n \to \infty$), then $\sum u_n$ converges provided $a > 1$ and diverges provided $a \leqslant 1$.

Proof. We have $\frac{u_n}{u_{n+1}} = 1 + \frac{a}{n} + \frac{b_n}{n^p}$.

$$\Rightarrow \quad \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = 1, \text{ as } |b_n| < k \text{ and } p > 1.$$

Hence D' Alembert's ratio test fails.

Now we shall apply Raabe's test.

$$\frac{u_n}{u_{n+1}} - 1 = \frac{a}{n} + \frac{n}{n^p}$$

$$\Rightarrow \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) = a + \frac{b_n}{n^{p-1}}$$

$$\Rightarrow \quad \lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = a, \text{ as } |b_n| < k \text{ and } p > 1.$$

Hence, by Raabe's test, $\sum u_n$ converges if $a > 1$ and diverges if $a < 1$.

If $a = 1$, then Raabe's test fails.

Now we shall apply De Morgan and Bertrand's test.

$$\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{b_n}{n^p}$$

$$\Rightarrow \frac{u_n}{u_{n+1}} - 1 = \frac{1}{n} + \frac{b_n}{n^p}$$

$$\Rightarrow n\left(\frac{u_n}{u_{n+1}} - 1\right) = 1 + \frac{b_n}{n^{p-1}}$$

$$\Rightarrow n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 = \frac{b_n}{n^{p-1}}$$

$$\Rightarrow \left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n = \frac{b_n \log n}{n^{p-1}}$$

$$\Rightarrow \lim_{n\to\infty} \left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n\right] = 0,$$

which is less than unity,

as $\lim_{n\to\infty} \frac{b_n \log n}{n^{p-1}} = 0$, (see note (b) below) for $|b_n| < k$ and $p > 1$.

Hence, by De Morgan and Bertrand's test, $\sum u_n$ is divergent.

Thus the proposition is established.

Note. (a) *Gauss's test can be enunciated as follows :*

(i) If $\sum u_n$ is a series of positive terms and if

$$\frac{u_{n+1}}{u_n} = 1 - \frac{\mu}{n} + \frac{A_n}{n^{1+\lambda}},$$

where μ is a constant, $\lambda > 0$ and A_n is bounded, then $\sum u_n$ converges whenever $\mu > 1$ and diverges otherwise.

(Bh U 1970 H; B U '58 H, '60 H; '63 H; P U '55 H)

Here $\frac{u_n}{u_{n+1}} = \frac{1}{1 - \frac{\mu}{n} + \frac{A_n}{n^{1+\lambda}}}$

$$= \left[1 - \left(\frac{\mu}{n} - \frac{A_n}{n^{1+\lambda}} \right) \right]^{-1}$$

$$= 1 + \frac{\mu}{n} - \frac{A_n}{n^{1+\lambda}} + \cdots$$

Now proceed as the proof given above.

(ii) Suppose that $u_n > 0$. If $\frac{u_n}{u_{n+1}}$ can be expanded as

$$\frac{u_n}{u_{n+1}} = \alpha + \frac{\beta}{n} + \frac{\mu_n}{n},$$

where α, β are constants, $\lambda > 1$, μ_n is finite for all values of n, then the series $\sum u_n$ is convergent, if $\alpha > 1$ and divergent if $\alpha < 1$.

If $\alpha = 1$, then the series is convergent, if $\beta > 1$ and divergent if $\beta \leqslant 1$. (R U 1966 H)

Here $\lim\limits_{n \to \infty} \frac{u_n}{u_{n+1}} = \alpha$.

Hence $\sum u_n$ converges when $\alpha > 1$ and diverges when $\alpha < 1$, by D' Alembert's ratio test.

If $\alpha = 1$, this test fails.

Now we shall apply Raabe's test.

Hence proceed as the proof given above.

(b) $\lim\limits_{n \to \infty} \frac{\log n}{n^k} = 0$ where $k > 0$, for we know

$$\lim_{n \to \infty} \frac{\log n}{n^k} = \lim_{x \to \infty} \frac{\log e^x}{(e^x)^k} = \lim_{x \to \infty} \frac{x}{e^{kx}}, \{\text{taking } n = e^x\}$$

and $$e^{kx} = 1 + kx + \frac{(kx)^2}{\lfloor\!\underline{2}} + \cdots + \frac{(kx)^m}{\lfloor\!\underline{m}} + \cdots$$

$$> \frac{(kx)^m}{\lfloor\!\underline{m}} \text{ where } m \text{ is } a \text{ fixed integer greater than } 1.$$

$$\therefore \quad \frac{e^{kx}}{x} > \frac{k^m}{\lfloor\!\underline{m}} x^{m-1}$$

If x increases indefinitely, x^{m-1} also increases indefinitely as m is a fixed integer > 1. So $\frac{e^{kx}}{x} \to \infty$ as $x \to \infty$.

Consequently $\frac{x}{e^{kx}} \to 0$ as $x \to \infty$.

3.13. Working Rule for selecting Tests to Examine Convergence of Series

By observation check whether the series is of positive terms.

(i) If so, then find $\lim_{n\to\infty} u_n$ if possible. If the limit is not zero the series is divergent.

(ii) If $\lim_{n\to\infty} u_n = 0$ apply comparison test. If not possible, apply the D' Alembert's Ratio test (specially in case of series containing powers of a variable).

(iii) In case $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$ apply Raabe's test or Logarithmic test in particular when n occurs as an exponent in terms.

(iv) If Raabe's test fails, apply De Morgan and Bertrand's test and if Logarithmic test fails, apply Higher Logarithmic test.

(v) If the terms are monotonic decreasing (particularly containing logarithmic function) Cauchy's condensation test should be tried.

(vi) If the series does not contain powers of a variable (or when D'Alembert's test fails), Gauss's test may be applied.

WORKED OUT EXAMPLES

1. *Examine the convergence of the following series.*

$$\Sigma \frac{(n+1)(n+2)}{(n+3)(n+4)}.$$

(Lucknow Univ 1961)

Solution. Let the given series be denoted by Σu_n.

Then $u_n = \frac{(n+1)(n+2)}{(n+3)(n+4)}$

and $u_{n+1} = \dfrac{(n+2)(n+3)}{(n+4)(n+5)}$.

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(n+1)(n+5)}{(n+3)^2} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{5}{n}\right)}{\left(1+\frac{3}{n}\right)^2}.$$

$$\therefore \quad \lim_{u\to\infty} \frac{u_n}{u_{n+1}} = 1.$$

Hence D'Alembert's ratio test fails.

Now we apply Raabe's test.

$$\frac{u_n}{u_{n+1}} - 1 = \frac{(n+1)(n+5)}{(n+3)^2} - 1 = \frac{n^2+6n+5-n^2-6n-9}{(n+3)^2}$$

$$\Rightarrow \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{-4n}{n^2+6n+9} = \frac{-\frac{4}{n}}{1+\frac{6}{n}+\frac{9}{n^2}}$$

$$\Rightarrow \quad \lim_{n\to\infty} \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right) \right\} = 0, \text{ which is less than 1.}$$

Hence, by Raabe's test, the given series is divergent.

Second Method :

$$\text{Here } u_n = \frac{(n+1)(n+2)}{(n+3)(n+4)} = \frac{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}{n^2\left(1+\frac{3}{n}\right)\left(1+\frac{4}{n}\right)}$$

$$= \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}{\left(1+\frac{3}{n}\right)\left(1+\frac{4}{n}\right)}$$

$\therefore \quad u_n \to 1 \neq 0$ when $n \to \infty$.

The necessary condition for a series to converge is that $u_n \to 0$ as $n \to \infty$. . Hence $\sum u_n$ is not convergent. But a series of positive terms will either converge or diverge. Hence $\sum u_n$ is divergent.

Third Method:

$$\text{Here } \frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{5}{n}\right)}{\left(1+\frac{3}{n}\right)^2}$$

$$= \left(1+\frac{6}{n}+\frac{5}{n^2}\right)\left(1+\frac{3}{n}\right)^{-2}$$

$$= \left(1+\frac{6}{n}+\frac{5}{n^2}\right)\left(1-2.\frac{3}{n}+3.\frac{9}{n^2}-\ldots\text{to}\,\infty\right)$$

$$= 1+\frac{0}{n}+\frac{1}{n^2}\left(-4\right)+\text{ terms containing } \frac{1}{n^3}, \frac{1}{n^4} \text{ etc.}$$

$$= 1+\frac{0}{n}+\frac{b_n}{n^2} \text{ where } b_n \to -4 \text{ as } n \to \infty.$$

$\therefore$ By Gauss's test, the series is divergent as $a = 0 < 1$.

Fourth Method:

$$\text{Here } u_n = \frac{(n+1)(n+2)}{(n+3)(n+4)} = \frac{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}{n^2\left(1+\frac{3}{n}\right)\left(1+\frac{4}{n}\right)}$$

$$= \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}{\left(1+\frac{3}{n}\right)\left(1+\frac{4}{n}\right)}$$

Let $v_n = 1$. Then $\lim_{n\to\infty} \frac{u_n}{v_n} = 1$ which is finite non-zero.

$\therefore$ $\sum u_n$ and $\sum v_n$ converge or diverge together.

But $\sum v_n = \sum 1 = \sum \frac{1}{n^0}$ is divergent as $p = 0 < 1$.

$\therefore$ $\sum u_n$ is divergent.

2. *Test the convergence of the series whose n^{th} term u_n is given by*

$$u_n = \frac{2^2.4^2.6^2\ldots.(2n-2)^2}{3.4.5\ldots.(2n-2)(2n-1)}\, x^{2n}. \qquad \text{(R U 1968 H)}$$

Solution. Here $u_n = \dfrac{2^2.4^2.6^2\ldots.(2n-2)^2}{3.4.5.\ldots.(2n-2)(2n-1)} x^{2n}$

and $u_{n+1} = \dfrac{2^2.4^2.6^2\ldots.(2n-2)^2.(2n)^2}{3.4.5\ldots.(2n-2)(2n-1)(2n)(2n+1)} x^{2n+2}$.

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{2n(2n+1)}{(2n)^2}.\frac{1}{x^2}$$

$$= \frac{2n+1}{2n}.\frac{1}{x^2} = \left(1+\frac{1}{2n}\right)\frac{1}{x^2}$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}.$$

Hence, by D' Alembert's ratio test, the given series is convergent or divergent according as

$$\frac{1}{x^2} > 1 \text{ or } \frac{1}{x^2} < 1$$

$\Rightarrow \quad x^2 < 1 \quad$ or $\quad x^2 > 1$.

If $x^2 = 1$, then this test fails.

Now we apply Raabe's test.

$$\therefore \quad \frac{u_n}{u_{n+1}} - 1 = \frac{2n+1}{2n} - 1 = \frac{1}{2n}$$

$$\Rightarrow \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{1}{2}$$

$$\Rightarrow \quad \lim_{n\to\infty} \left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = \frac{1}{2}, \text{ which is less than unity.}$$

Hence, by Raabe's test, the given series is divergent when $x^2 = 1$.

3. *Test the convergence of the series*

$$\frac{x}{2} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^3}{6} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{x^5}{10} + \ldots \text{to } \infty.$$

(P U 1950 H; B U '53 H)

Solution. Let the given series be denoted by $\sum u_n$. Without loss of generality we can assume $x > 0$ for if x is negative, all the terms will be negative and so by taking negative sign common it will be a series of positive terms.

Then $u_n = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{4n-5}{4n-4} \cdot \frac{x^{2n-1}}{4n-2}$, and

$$u_{n+1} = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{4n-5}{4n-4} \cdot \frac{4n-3}{4n-2} \cdot \frac{4n-1}{4n} \cdot \frac{x^{2n+1}}{4n+2}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{4n-2}{4n-3} \cdot \frac{4n}{4n-1} \cdot \frac{4n+2}{4n-2} \cdot \frac{1}{x^2}$$

$$= \frac{4n(4n+2)}{(4n-3)(4n-1)} \cdot \frac{1}{x^2}$$

$$= \frac{\left(1 + \frac{1}{2n}\right)}{\left(1 - \frac{3}{4n}\right)\left(1 - \frac{1}{4n}\right)} \cdot \frac{1}{x^2}$$

$$\Rightarrow \quad \lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}.$$

Hence, by D'Alembert's ratio test, the given series is convergent or divergent according as

$$\frac{1}{x^2} > 1 \quad \text{or} \quad \frac{1}{x^2} < 1$$

$\Rightarrow \quad x^2 < 1 \quad \text{or} \quad x^2 > 1$

$\Rightarrow \quad |x| < 1 \quad \text{or} \quad |x| > 1.$

When $x^2 = 1$, then this test fails.

Now we apply Raabe's test.

$$\therefore \quad \frac{u_n}{u_{n+1}} - 1 = \frac{4n(4n+2)}{(4n-3)(4n-1)} - 1$$

$$= \frac{16n^2 + 8n - 16n^2 + 16n - 3}{(4n-3)(4n-1)}$$

$$= \frac{24n - 3}{(4n-3)(4n-1)}$$

$$\Rightarrow \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{24n^2 - 3n}{(4n-3)(4n-1)}$$

$$= \frac{24 - \frac{3}{n}}{\left(4 - \frac{3}{n}\right)\left(4 - \frac{1}{n}\right)}$$

$$\Rightarrow \lim_{n\to\infty} \left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = \frac{24}{16} = \frac{3}{2},$$

which is greater than unity.

Hence, by Raabe's test, the given series is convergent when

$$x^2 = 1 \Rightarrow |x| = 1.$$

4. *Test the convergence of the series*

$$1 + a + \frac{a(a+1)}{1.2} + \frac{a(a+1)(a+2)}{1.2.3} + \ldots \textit{to } \infty.$$

Solution. Clearly for a given value of a (positive or negative) it is a series of terms of the same sign after some stage. Omitting the first term of the given series, as this does not affect the convergence or the divergence of the series and denoting the series by $\sum u_n$, we get

$$u_n = \frac{a(a+1)(a+2)\ldots(a+n-1)}{1.2.3\ldots n} \text{ and}$$

$$u_{n+1} = \frac{a(a+1)(a+2)\ldots(a+n-1)(a+n)}{1.2.3\ldots n(n+1)}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{n+1}{a+n} = \frac{1 + \frac{1}{n}}{\frac{a}{n} + 1}; \quad \therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1.$$

Hence D'Alembert's ratio test fails.

Now we apply Raabe's test

$$\frac{u_n}{u_{n+1}} - 1 = \frac{n+1}{a+n} - 1 = \frac{1-a}{a+n}$$

$$\Rightarrow \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{1-a}{\frac{a}{n} + 1}$$

$$\Rightarrow \quad \lim_{n\to\infty} \left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = 1 - a.$$

Hence, by Raabe's test, the given series is convergent or divergent according as $1 - a > 1$ or $1 - a < 1$

$$\Rightarrow \; a < 0 \;\text{ or }\; a > 0.$$

If $a = 0$, then the given series $= 1 + 0 + 0 + \dots = 1$.

Hence the given series is convergent when $a = 0$.

5. *Test the convergence for positive real values of x—*

$$x^2(\log_e 2)^p + x^3(\log_e 3)^p + x^4(\log_e 4)^p + \dots \text{ to } \infty.$$

(P U 1961 H; B U '61 H, '64 H; R U '62 H, '64 H; M U '73 H)

Solution. Let the given series be denoted by $\sum u_n$.

Then $u_n = x^{n+1}\{\log(n+1)\}^p$ and $u_{n+1} = x^{n+2}\{\log(n+2)\}^p$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{1}{x} \cdot \left\{\frac{\log(n+1)}{\log(n+2)}\right\}^p$$

$$= \frac{1}{x} \cdot \left[\frac{\log\{(n+2)-1\}}{\log(n+2)}\right]^p$$

$$= \frac{1}{x} \cdot \left[\frac{\log\left\{(n+2)\left(1 - \frac{1}{n+2}\right)\right\}}{\log(n+2)}\right]^p$$

$$= \frac{1}{x} \cdot \left[\frac{\log(n+2) + \log\left(1 - \frac{1}{n+2}\right)}{\log(n+2)}\right]^p$$

$$= \frac{1}{x} \cdot \left[1 - \frac{1}{(n+2)\log(n+2)} - \frac{1}{2(n+2)^2\log(n+2)} - \dots\right]^p$$

$$= \frac{1}{x} \cdot \left[1 - \frac{p}{(n+2)\log(n+2)} - \frac{p}{2(n+2)^2\log(n+2)} \dots\right],$$

by Binomial theorem

$$\Rightarrow \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

Hence, by D'Alembert's ratio test, the given series is convergent or divergent according as $x < 1$ or $x > 1$.

If $x = 1$ then this test fails.

Now we apply Raabe's test

$$\frac{u_n}{u_{n+1}} = 1 - \frac{p}{(n+2)\log(n+2)} - \frac{p}{2(n+2)^2\log(n+2)} - \cdots$$

$$\Rightarrow \quad n\left(\frac{u_n}{u_{n+1}} - 1\right)$$

$$= -\frac{p}{\left(1+\frac{2}{n}\right)\log(n+2)} - \frac{\frac{p}{n}}{2\left(1+\frac{2}{n}\right)^2\log(n+2)} - \cdots$$

$$\Rightarrow \quad \lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = 0, \text{ which is less than 1.}$$

Hence, by Raabe's test, the given series is divergent, when $x = 1$.

6. *Test the convergence of* $\sum_1^\infty \left(\frac{nx}{n+1}\right)^n$, $(x > 0)$.

(B U 1959 H)

Solution. Let the given series be denoted by $\sum u_n$.

Then $u_n = \left(\frac{nx}{n+1}\right)^n = \frac{n^n x^n}{(n+1)^n}$ and

$$u_{n+1} = \frac{(n+1)^{n+1}x^{n+1}}{(n+2)^{n+1}}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(n+2)^{n+1}}{(n+1)^n}\cdot\frac{n^n}{(n+1)^{n+1}}\cdot\frac{1}{x}$$

$$= \frac{(n+1+1)^{n+1}}{(n+1)^{n+1}}\cdot\left(\frac{n}{n+1}\right)^n\cdot\frac{1}{x}$$

$$= \left(1+\frac{1}{n+1}\right)^{n+1}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{x}$$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{e}{e}\cdot\frac{1}{x} = \frac{1}{x}.$$

Hence, by D'Alembert's ratio test, the given series is convergent or divergent according as

$$\frac{1}{x} > 1 \quad \text{or} \quad \frac{1}{x} < 1$$

$\Rightarrow \quad x < 1 \quad \text{or} \quad x > 1.$

When $x = 1$, this test fails.

Now we apply logarithmic ratio test.

$$\therefore \quad \log \frac{u_n}{u_{n+1}} = \log \left(1 + \frac{1}{n+1}\right)^{n+1} - \log \left(1 + \frac{1}{n}\right)^n$$

$$= (n+1)\left[\frac{1}{n+1} - \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} - \ldots \text{to } \infty\right] - n\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots \text{to } \infty\right]$$

$$= \left[-\frac{1}{2(n+1)} + \frac{1}{3(n+1)^2} - \ldots \text{to } \infty\right] + \left[\frac{1}{2n} - \frac{1}{3n^2} + \ldots \text{to } \infty\right]$$

$$\Rightarrow \quad n \log \frac{u_n}{u_{n+1}}$$

$$= \left[-\frac{1}{2\left(1 + \frac{1}{n}\right)} + \frac{\frac{1}{n}}{3\left(1 + \frac{1}{n}\right)^2} - \ldots \text{to } \infty\right] + \left[\frac{1}{2} - \frac{1}{3n} + \ldots \text{to } \infty\right]$$

$$\Rightarrow \quad \lim_{n \to \infty} \left\{n \log \frac{u_n}{u_{n+1}}\right\} = -\frac{1}{2} + \frac{1}{2} = 0,$$

which is less than unity.

Hence, by logarithmic ratio test, the given series is divergent when $x = 1$.

7. *Examine the convergence for* $x > 0$—

$$x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \ldots \textit{to } \infty.$$

(B P S C 1978; Lucknow U '46; Rajputana U '58;
Agra U '63; Roorkee Engg. '79)

Solution. Let the given series be denoted by $\sum u_n$.

Then $u_n = \dfrac{n^n x^n}{n!}$ and $u_{n+1} = \dfrac{(n+1)^{n+1}x^{n+1}}{(n+1)!}$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(n+1)!}{n!} \cdot \frac{n^n x^n}{(n+1)^{n+1}x^{n+1}}$$

$$= \frac{n!\,(n+1)}{n!} \cdot \frac{n^n}{(n+1)(n+1)^n} \cdot \frac{1}{x}$$

$$= \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{x} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{x}$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{e.x}, \text{ as } \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

Hence, by D'Alembert's ratio test, the given series is convergent or divergent according as

$$\frac{1}{ex} > 1 \quad \text{or} \quad \frac{1}{ex} < 1$$

$$\Rightarrow \quad x < \frac{1}{e} \quad \text{or} \quad x > \frac{1}{e}.$$

If $x = \dfrac{1}{e}$ then this test fails.

Now we apply logarithmic ratio test

$$\frac{u_n}{u_{n+1}} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot e$$

$$\Rightarrow \quad \log \frac{u_n}{u_{n+1}} = \log e - n \log\left(1+\frac{1}{n}\right)$$

$$= 1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots \text{to } \infty\right)$$

$$\Rightarrow \quad n \log \frac{u_n}{u_{n+1}} = \frac{1}{2} - \frac{1}{3n} + \ldots \text{to } \infty$$

$$\Rightarrow \quad \lim_{n\to\infty}\left(n \log \frac{u_n}{u_{n+1}}\right) = \frac{1}{2}, \text{ which is less than 1.}$$

Hence, by logarithmic ratio test, the given series is divergent when $x = \frac{1}{e}$.

8. *Test the convergence of the series when* $x > 0$—

$$\frac{a+x}{1\,!} + \frac{(a+2x)^2}{2\,!} + \frac{(a+3x)^3}{3\,!} + \ldots\ldots \textit{to } \infty.$$

Solution. Clearly for all values of a, it is a series of positive terms after some stage. Let the given series be denoted by $\sum u_n$.

Then $u_n = \frac{(a+nx)^n}{n\,!}$ and $u_{n+1} = \frac{\{a+(n+1)x\}^{n+1}}{(n+1)\,!}$.

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(a+nx)^n}{\{a+(n+1)x\}^{n+1}} \cdot \frac{(n+1)\,!}{n\,!}$$

$$= \frac{n^n x^n \left(1 + \frac{a}{nx}\right)^n}{(n+1)^{n+1} x^{n+1} \left\{1 + \frac{a}{(n+1)x}\right\}^{n+1}} \times (n+1)$$

$$= \left(\frac{1}{1+\frac{1}{n}}\right)^n \cdot \frac{1}{x} \cdot \frac{\left\{\left(1 + \frac{1}{\frac{nx}{a}}\right)^{\frac{nx}{a}}\right\}^{\frac{a}{x}}}{\left[\left\{1 + \frac{1}{\frac{(n+1)x}{a}}\right\}^{(n+1)\frac{x}{a}}\right]^{\frac{a}{x}}}$$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{e} \cdot \frac{1}{x} \cdot \frac{e^{\frac{a}{x}}}{e^{\frac{a}{x}}} = \frac{1}{ex}.$$

Hence, by D'Alembert's ratio test, the given series is convergent or divergent according as

$$\frac{1}{ex} > 1 \quad \text{or} \quad \frac{1}{ex} < 1$$

$$\Rightarrow \quad x < 1/e \quad \text{or} \quad x > 1/e.$$

If $\frac{1}{ex} = 1$ i.e. $x = \frac{1}{e}$ then this test fails.

Now we apply logarithmic ratio test

$$\frac{u_n}{u_{n+1}} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{\left(1+\frac{ae}{n}\right)^n}{\left(1+\frac{ae}{n+1}\right)^{n+1}} \cdot e$$

$$\Rightarrow \log\frac{u_n}{u_{n+1}} = \log e + n\log\left(1+\frac{ae}{n}\right) - n\log\left(1+\frac{1}{n}\right) - (n+1)\log\left(1+\frac{ae}{n+1}\right)$$

$$= 1 + n\left(\frac{ae}{n} - \frac{1}{2}\frac{a^2e^2}{n^2} + \frac{1}{3}\frac{a^3e^3}{n^3} - \dots\right) - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots\right)$$

$$- (n+1)\left[\frac{ae}{n+1} - \frac{1}{2}\frac{a^2e^2}{(n+1)^2} + \frac{1}{3}\frac{a^3e^3}{(n+1)^3} - \dots\right]$$

$$\Rightarrow n\log\frac{u_n}{u_{n+1}} = \left(-\frac{1}{2}a^2e^2 + \frac{1}{3}\frac{a^3e^3}{n} - \dots\right) - \left(-\frac{1}{2} + \frac{1}{3n} - \dots\right)$$

$$- \left[-\frac{1}{2}\frac{a^2e^2}{1+\frac{1}{n}} + \frac{1}{3}\frac{a^3e^3}{n\left(1+\frac{1}{n}\right)^2} - \dots\right]$$

$$\Rightarrow \lim_{n\to\infty} n\log\frac{u_n}{u_{n+1}} = -\frac{1}{2}a^2e^2 + \frac{1}{2} + \frac{1}{2}a^2e^2 = \frac{1}{2},$$

which is less than one.

Hence, by logarithmic ratio test, the given series is divergent when $x = \frac{1}{e}$.

9. *Test the convergence of the series whose n^{th} term is given by*

$$u_n = \left\{\frac{1.3.5....(2n-1)}{2.4.6....2n}\right\}^2\left(\frac{n}{n+1}\right).$$

(B U 1960 H)

Solution. Here $u_n = \left\{\frac{1.3.5....(2n-1)}{2.4.6....2n}\right\}^2\left(\frac{n}{n+1}\right).$

$$\therefore \quad u_{n+1} = \left\{\frac{1.3.5....(2n-1)(2n+1)}{2.4.6....2n(2n+2)}\right\}^2\left(\frac{n+1}{n+2}\right).$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \left(\frac{2n+2}{2n+1}\right)^2 \cdot \frac{n(n+2)}{(n+1)^2}$$

$$= \frac{4(n+1)^2}{(2n+1)^2} \cdot \frac{n(n+2)}{(n+1)^2} = \frac{4n(n+2)}{4n^2+4n+1}$$

$$= \frac{4\left(1+\frac{2}{n}\right)}{4+\frac{4}{n}+\frac{1}{n^2}}$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1.$$

Hence D'Alembert's ratio test fails.

Now we apply Raabe's test

$$\therefore \quad \frac{u_n}{u_{n+1}} - 1 = \frac{4n^2+8n}{4n^2+4n+1} - 1 = \frac{4n-1}{4n^2+4n+1}$$

$$\Rightarrow \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{4n^2-n}{4n^2+4n+1} = \frac{4-\frac{1}{n}}{4+\frac{4}{n}+\frac{1}{n^2}}$$

$$\Rightarrow \quad \lim_{n\to\infty} \left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = 1.$$

Hence Raabe's test fails.

Now we apply De Morgan and Bertrand's test

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 = \frac{4n^2-n}{4n^2+4n+1} - 1 = \frac{-5n-1}{4n^2+4n+1}$$

$$\Rightarrow \left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n = \frac{-5n^2 - n}{4n^2 + 4n + 1} \cdot \frac{\log n}{n}$$

$$= \frac{-5 - \dfrac{1}{n}}{4 + \dfrac{4}{n} + \dfrac{1}{n^2}} \cdot \frac{\log n}{n}$$

$$\Rightarrow \lim_{n\to\infty} \left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n\right] = -\frac{5}{4} \times 0 = 0,$$

which is less than unity.

$$\left(\because \lim_{n\to\infty} \frac{\log n}{n} = 0\right)$$

Hence, by De Morgan and Bertrand's test, the given series is divergent.

10. *Test the convergence of the series* $(x > 0)$—

$$1 + \frac{2^2}{3^2}x + \frac{2^2.4^2}{3^2.5^2}x^2 + \frac{2^2.4^2.6^2}{3^2.5^2.7^2}x^3 + \ldots. \textit{to } \infty.$$

(M U 1965 H; P U '73 H)

Solution. Leaving the first term of the given series as this does not affect the convergence or the divergence of the series and then denoting the series by $\sum u_n$, we get

$$u_n = \frac{2^2.4^2.6^2\ldots.(2n)^2}{3^2.5^2.7^2\ldots.(2n+1)^2}x^n, \text{ and}$$

$$u_{n+1} = \frac{2^2.4^2.6^2\ldots.(2n)^2(2n+2)^2}{3^2.5^2.7^2\ldots.(2n+1)^2(2n+3)^2}x^{n+1}.$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(2n+3)^2}{(2n+2)^2} \cdot \frac{1}{x} = \frac{\left(1 + \dfrac{3}{2n}\right)^2}{\left(1 + \dfrac{1}{n}\right)^2} \cdot \frac{1}{x}$$

$$\Rightarrow \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

Hence, by D'Alembert's ratio test, the given series is convergent or divgerent according as

$$\frac{1}{x} > 1 \quad \text{or} \quad \frac{1}{x} < 1$$

$$\Rightarrow \quad x < 1 \quad \text{or} \quad x > 1.$$

If $x = 1$, then this test fails.

Now we apply Raabe's test.

$$\therefore \quad \frac{u_n}{u_{n+1}} - 1 = \frac{(2n+3)^2}{(2n+2)^2} - 1 = \frac{4n+5}{(2n+2)^2}$$

$$\Rightarrow \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{4n^2+5n}{(2n+2)^2} = \frac{1+\dfrac{5}{4n}}{\left(1+\dfrac{1}{n}\right)^2}$$

$$\Rightarrow \quad \lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = 1.$$

Hence Raabe's test fails.

Now we apply De Morgan and Bertrand's test.

$$\therefore \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 = \frac{4n^2+5n}{(2n+2)^2} - 1 = \frac{-3n-4}{(2n+2)^2}$$

$$\Rightarrow \quad \left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\}\log n = \frac{-3n^2-4n}{(2n+2)^2}\cdot\frac{\log n}{n}$$

$$= \frac{-3-\dfrac{4}{n}}{\left(2+\dfrac{2}{n}\right)^2}\cdot\frac{\log n}{n}$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\}\log n\right] = -\frac{3}{4}\times 0 = 0,$$

which is less than unity.

$$\left(\because \lim_{n\to\infty}\frac{\log n}{n} = 0\right)$$

Hence, by De Morgan and Bertrand's test, the given series is divergent, when $x = 1$.

11. *Prove that*

$$1 + \frac{1}{2}\cdot\frac{a}{b} + \frac{1.3}{2.4}\cdot\frac{a(a+1)}{b(b+1)} + \frac{1.3.5\ a(a+1)(a+2)}{2.4.6\ b(b+1)(b+2)} + \ldots to\ \infty$$

is convergent if $b > a + \frac{1}{2}$. *What happens when* $b = a + \frac{1}{2}$?

(M U 1963 H)

Solution. Leave the first term as this does not affect the convergence or the divergence of the series.

Let the given series be denoted by $\sum u_n$.

Clearly for given values of a and b, it is a series of positive terms after some stage. Now

$$u_n = \frac{1.3.5\ldots.(2n-1)}{2.4.6\ldots.2n} \cdot \frac{a(a+1)(a+2)\ldots.(a+n-1)}{b(b+1)(b+2)\ldots.(b+n-1)}$$

and u_{n+1}

$$= \frac{1.3.5..(2n-1)(2n+1)}{2.4.6..(2n)(2n+2)} \cdot \frac{a(a+1)\ldots.(a+n-1)(a+n)}{b(b+1)\ldots.(b+n-1)(b+n)}$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1} \cdot \frac{b+n}{a+n} = \frac{1+\dfrac{1}{n}}{1+\dfrac{1}{2n}} \cdot \frac{1+\dfrac{b}{n}}{1+\dfrac{a}{n}}$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1.$$

Hence D'Alembert's ratio test fails.

Now we apply Raabe's test.

$$\therefore \quad \frac{u_n}{u_{n+1}} - 1 = \frac{(2n+2)(b+n)}{(2n+1)(a+n)} - 1$$

$$= \frac{2nb + 2n^2 + 2b + 2n - 2na - 2n^2 - a - n}{(2n+1)(a+n)}$$

$$= \frac{n(2b+1-2a) + (2b-a)}{(2n+1)(a+n)}$$

$$\Rightarrow \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{n^2(2b+1-2a) + n(2b-a)}{(2n+1)(a+n)}$$

$$= \frac{2b+1-2a+\dfrac{1}{n}(2b-a)}{\left(2+\dfrac{1}{n}\right)\left(\dfrac{a}{n}+1\right)}$$

$$\Rightarrow \quad \lim_{n\to\infty} \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right) \right\} = \frac{2b+1-2a}{2}.$$

Hence, by Raabe's test, the given series is convergent or divergent according as

$$\frac{2b+1-2a}{2} > 1 \quad \text{or} \quad \frac{2b+1-2a}{2} < 1$$

$$\Rightarrow \quad 2b+1-2a > 2 \quad \text{or} \quad 2b+1-2a < 2$$

$$\Rightarrow \quad 2b > 1+2a \quad \text{or} \quad 2b < 1+2a$$

$$\Rightarrow \quad b > a+\tfrac{1}{2} \quad \text{or} \quad b < a+\tfrac{1}{2}.$$

If $b = a + \frac{1}{2}$ then this test fails.

Now we apply De Morgan and Bertrand's test.

$$\therefore \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 = \frac{2n^2 + n(2b-a)}{(2n+1)(a+n)} - 1$$

$$= \frac{2n^2 + n(2b-a) - 2na - 2n^2 - a - n}{(2n+1)(a+n)}$$

$$= \frac{n(2b-a-2a-1) - a}{(2n+1)(a+n)}$$

$$= \frac{-an-a}{(2n+1)(a+n)}$$

$$\Rightarrow \quad \left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\}\log n = \frac{-an^2 - an}{(2n+1)(a+n)} \cdot \frac{\log n}{n}$$

$$= \frac{-a - \dfrac{a}{n}}{\left(2 + \dfrac{1}{n}\right)\left(1 + \dfrac{a}{n}\right)} \cdot \frac{\log n}{n}$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\}\log n\right] = -\frac{a}{2} \times 0 = 0,$$

which is less than 1.

$$\left(\because \quad \lim_{n\to\infty}\frac{\log n}{n} = 0\right)$$

Hence, by De Morgan and Bertrand's test, the given series is divergent when $b = a + \frac{1}{2}$.

12. *Examine the convergence for* $x > 0$—

$$\frac{a}{b} + \frac{a(a+1)}{b(b+1)}x + \ldots + \frac{a(a+1)(a+2)\ldots(a+n)}{b(b+1)(b+2)\ldots(b+n)}x^n + \ldots$$

(P U 1959)

Solution. Here $u_n = \dfrac{a(a+1)\ldots(a+n-1)}{b(b+1)\ldots(b+n-1)}x^{n-1}$.

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{b+n}{a+n}\cdot\frac{1}{x} = \frac{1+\dfrac{b}{n}}{1+\dfrac{a}{n}}\cdot\frac{1}{x}.$$

$$\therefore \quad \operatorname*{Lt}_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

$\therefore$ If $x < 1$, $\operatorname*{Lt}_{n\to\infty} \dfrac{u_n}{u_{n+1}} > 1$ and the series is convergent.

If $x > 1$, $\operatorname*{Lt}_{n\to\infty} \dfrac{u_n}{u_{n+1}} < 1$ and the series is divergent.

If $x = 1$, and $a \geqslant b$, $\dfrac{u_n}{u_{n+1}} \leqslant 1$, i.e., $u_{n+1} \geqslant u_n$ for all n.

As the series is of positive terms and

$$u_{n+1} \geqslant u_n \geqslant u_{n-1} \geqslant \ldots \geqslant u_1,$$

$\therefore$ $u_1 + u_2 + \ldots \geqslant nu_1$ where $n \to \infty$.

But $nu_1 \to \infty$ as $n \to \infty$.

$\therefore$ Σu_n is divergent.

If $x = 1$ and $a < b$,

$$\frac{u_n}{u_{n+1}} - 1 = \frac{b+n}{a+n} - 1 = \frac{b-a}{a+n}.$$

$$\therefore \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{n}{a+n}\cdot(b-a);$$

$$\therefore \quad \operatorname*{Lt}_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = b - a.$$

So if $b > a + 1$, $\operatorname*{Lt}_{n\to\infty} n\left(\dfrac{u_n}{u_{n+1}} - 1\right) > 1$ and so by Raabe's Test the series is convergent and if $b < a + 1$, the limit is less than one and hence the series is divergent.

If $b = a + 1$ we go to De Morgan and Bertrand's test.

Now $n\left(\dfrac{u_n}{u_{n+1}} - 1\right) - 1 = \dfrac{n}{a+n} - 1 = \dfrac{-a}{a+n}$

$$\Rightarrow \lim_{n\to\infty}\left[\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n\right]$$

$$= -\lim_{n\to\infty}\frac{a}{1+\dfrac{a}{n}}\cdot\lim_{n\to\infty}\frac{\log n}{n}$$

$$= -a\times 0 = 0 < 1.$$

Hence, by De Morgan and Bertrand's test the given series is divergent when $b = a+1$.

13. *Examine the convergence of the series* $(x > 0)$.

$$1+\frac{\alpha\beta}{1\,.\,\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\,.\,2\,.\,\gamma(\gamma+1)}x^2$$

$$+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\,.\,2\,.\,3\,.\,\gamma(\gamma+1)(\gamma+2)}x^3+\ldots\ to\ \infty,\ (x>0).$$

(Sagar U 1949; Allahabad U '39; Roorkee Engg '71;
P U '49 H; B U '55 H, '57 H, '59 H, '68 H;
Bh U '65 H, '74 H; M U '70 H)

Note. The above series is known as hypergeometric series.

Solution. Clearly it is a series of positive terms after some stage for all α, β, γ. Omitting the first term of the given series as this does not affect the convergence or the divergence of the series and now denoting the series by Σu_n, we get

$$u_n = \frac{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1)\beta(\beta+1)(\beta+2)\ldots(\beta+n-1)}{1\,.\,2\,.\,3\ldots n.\gamma(\gamma+1)(\gamma+2)\ldots(\gamma+n-1)}x^n$$

and

$$u_{n+1} = \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)(\alpha+n)\beta(\beta+1)\ldots(\beta+n-1)(\beta+n)}{1.2\ldots n(n+1)\gamma(\gamma+1)\ldots(\gamma+n-1)(\gamma+n)}x^{n+1}.$$

$$\therefore\ \frac{u_n}{u_{n+1}} = \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)}\cdot\frac{1}{x} = \frac{\left(1+\dfrac{1}{n}\right)\left(1+\dfrac{\gamma}{n}\right)}{\left(1+\dfrac{\alpha}{n}\right)\left(1+\dfrac{\beta}{n}\right)}\cdot\frac{1}{x}$$

$$\Rightarrow \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{x}.$$

Hence, by D'Alembert's ratio test, the given series is convergent or divergent according as

$$\frac{1}{x} > 1 \quad \text{or} \quad \frac{1}{x} < 1$$

$$\Rightarrow \quad x < 1 \quad \text{or} \quad x > 1.$$

If $\frac{1}{x} = 1$ i.e. $x = 1$ then this test fails.

Now we apply Raabe's test.

$$\frac{u_n}{u_{n+1}} - 1 = \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)} - 1$$

$$= \frac{n\gamma + n^2 + \gamma + n - \alpha\beta - \alpha n - \beta n - n^2}{(\alpha+n)(\beta+n)}$$

$$\Rightarrow \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{n^2(1+\gamma-\alpha-\beta) + (\gamma - \alpha\beta)n}{(\alpha+n)(\beta+n)}$$

$$= \frac{1 + \gamma - \alpha - \beta + \dfrac{\gamma - \alpha\beta}{n}}{\left(1 + \dfrac{\alpha}{n}\right)\left(1 + \dfrac{\beta}{n}\right)}$$

$$\Rightarrow \quad \lim_{n\to\infty} \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right) \right\} = 1 + \gamma - \alpha - \beta.$$

Hence, by Raabe's test, the given series is convergent or divergent according as

$$1 + \gamma - \alpha - \beta > 1 \quad \text{or} \quad 1 + \gamma - \alpha - \beta < 1$$

$$\Rightarrow \quad \gamma > \alpha + \beta \quad \text{or} \quad \gamma < \alpha + \beta.$$

If $1 + \gamma - \alpha - \beta = 1$ i.e. $\gamma = \alpha + \beta$, this test fails.

Now we apply De Morgan and Bertrand's test.

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 = \frac{n^2 + n(\gamma - \alpha\beta)}{(\alpha+n)(\beta+n)} - 1$$

$$= \frac{n^2 + n(\gamma - \alpha\beta) - \alpha\beta - \alpha n - \beta n - n^2}{(\alpha+n)(\beta+n)}$$

$$= \frac{n(\gamma - \alpha\beta - \alpha - \beta) - \alpha\beta}{(\alpha+n)(\beta+n)}$$

$$= \frac{-n\,\alpha\beta - \alpha\beta}{(\alpha + n)(\beta + n)}, \{\because\ \gamma = \alpha + \beta\}$$

$$\Rightarrow \quad \left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n = \frac{-n\,\alpha\beta - \alpha\beta}{(\alpha + n)(\beta + n)} \cdot \log n$$

$$= \frac{-n^2\alpha\beta - n\,\alpha\beta}{(\alpha + n)(\beta + n)} \cdot \frac{\log n}{n}$$

$$= \frac{-\alpha\beta - \dfrac{\alpha\beta}{n}}{\left(1 + \dfrac{\alpha}{n}\right)\left(1 + \dfrac{\beta}{n}\right)} \cdot \frac{\log n}{n}$$

$$\Rightarrow \quad \lim_{n\to\infty} \left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n\right] = -\alpha\beta.0 = 0 < 1.$$

$$\left(\because\ \lim_{n\to\infty} \frac{\log n}{n} = 0\right)$$

Hence, by De Morgan and Bertrand's test, the given series is divergent when $\gamma = \alpha + \beta$.

Note. The case of $x = 1$ can also be examined by Gauss' test as $\frac{u_n}{u_{n+1}} = \left(1 + \frac{1}{n}\right)\left(1 + \frac{\gamma}{n}\right)\left(1 + \frac{\alpha}{n}\right)^{-1}\left(1 + \frac{\beta}{n}\right)^{-1}$

$$= 1 + \frac{\gamma + 1 - \alpha - \beta}{n} + 0\left(\frac{1}{n^2}\right).$$

14. *If* $\frac{u_n}{u_{n+1}} = \frac{n^k + An^{k-1} + Bn^{k-2} + \cdots\cdots}{n^k + an^{k-1} + bn^{k-2} \quad \cdots\cdots}$; *where* k *is a positive integer, show that the series* $\sum u_n$ *is convergent if* $A - a - 1$ *is positive and divergent if* $A - a - 1$ *is negative or zero.*

Solution. We have $\frac{u_n}{u_{n+1}} = \frac{n^k + An^{k-1} + Bn^{k-2} + \cdots}{n^k + an^{k-1} + bn^{k-2} + \cdots}$

$$= \frac{1 + A/n + B/n^2 + \cdots}{1 + a/n + b/n^2 + \cdots}$$

$\Rightarrow \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1$. Hence D' Alembert's ratio test fails.

Now we apply Raabe's test.

$$\frac{u_n}{u_{n+1}} - 1 = \frac{(A-a)n^{k-1} + (B-b)n^{k-2} + \ldots.}{n^k + an^{k-1} + bn^{k-2} + \ldots.}$$

$$\Rightarrow \quad n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{(A-a)n^k + (B-b)n^{k-1} + \ldots.}{n^k + an^{k-1} + bn^{k-2} + \ldots.}$$

$$= \frac{(A-a) + \frac{1}{n}(B-b) + \ldots.}{1 + \frac{a}{n} + \frac{b}{n^2} + \ldots.}$$

$$\Rightarrow \quad \lim_{n\to\infty} \left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = A - a.$$

Hence, by Raabe's test, the given series is convergent or divergent according as $A - a > 1$ or $A - a < 1$.

i.e. $\quad A - a - 1 > 0$ or $A - a - 1 < 0$

If $A - a = 1$, this test fails.

Now we apply. De Morgan and Bertrand's test.

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 = \frac{n^k + (B-b)n^{k-1} + \ldots.}{n^k + an^{k-1} + bn^{k-2} + \ldots.} - 1$$

$$= \frac{(B-b-a)n^{k-1} + \ldots.}{n^k + an^{k-1} + bn^{k-2} + \ldots.}$$

$$\therefore \quad \left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\}\log n = \frac{n^k(B-b-a) + \ldots.}{n^k + an^{k-1} + \ldots.} \cdot \frac{\log n}{n}$$

$$= (B-b-a)\times 0$$

$$= 0, \left(\because \lim_{n\to\infty} \frac{\log n}{n} = 0\right)$$

which is less than 1.

Hence, by De Morgan and Bertrand's test the given series is divergent, when $A - a = 1$, $A - a - 1$ is zero.

15. *Test for convergence the series whose general term is*

$$\frac{n^{-1}}{\sqrt{\log n}}.$$

Solution. The given series is $\sum \dfrac{n^{-1}}{\sqrt{\log n}}$,

$$\text{that is, } \sum \frac{1}{n\,(\log n)^{\frac{1}{2}}}.$$

We know that $\sum \dfrac{1}{n\,(\log n)^p}$ is convergent when $p > 1$ and divergent when $p \leqslant 1$.

Here $p = \frac{1}{2}$ which is less than 1.

Hence the given series is divergent.

16. *Test for convergence the series*

$$\sum_{n=2}^{\infty} \frac{\log n}{n^p} \qquad \text{(M U 1967 H)}$$

Solution. Let the given series be denoted by $\sum f(n)$.

$$\text{Then } f(n) = \frac{\log n}{n^p}.$$

CASE I. Let $p \leqslant 0$ and $p = -m$ where $m \geqslant 0$

Then $\lim\limits_{n\to\infty} f(n) = \lim\limits_{n\to\infty} (\log n . n^m) = \infty$.

Hence the given series $\sum f(n)$ is divergent.

CASE II. Let $p > 0$.

Then $f(n) > 0$ and monotonic decreasing, as n^p, $(p > 0)$ increases at a faster rate than $\log n$ as n increases.

Let a be a positive integer greater than unity.

$$\text{Now } a^n f(a^n) = a^n \frac{\log a^n}{(a^n)^p} = \frac{n \log a}{a^{n(p-1)}} = V_n(\text{say}).$$

$$\therefore \quad V_{n+1} = \frac{(n+1)\log a}{a^{(n+1)(p-1)}}$$

$$\therefore \quad \frac{V_n}{V_{n+1}} = \frac{a^{p-1}}{1 + \dfrac{1}{n}}$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{V_n}{V_{n+1}} = a^{p-1}$$

Hence, by D' Alembert's ratio test, the series $\sum V_n$, that is, $\sum a^n f(a^n)$ is convergent or divergent according as

$$\begin{aligned} & a^{p-1} > 1 \text{ or } a^{p-1} < 1 \\ \Rightarrow\ & a^{p-1} > a^o \text{ or } a^{p-1} < a^o \\ \Rightarrow\ & p-1 > 0 \text{ or } p-1 < 0 \\ \Rightarrow\ & p > 1 \quad \text{ or } p < 1. \end{aligned}$$

This test fails when $p = 1$.

In this case $a^n f(a^n) = n \log a$

$$\Rightarrow \lim_{n\to\infty} a^n f(a^n) = \infty, \ \{\because a > 1\}$$

Therefore $\sum a^n f(a^n)$ is divergent when $p = 1$.

Hence, by Cauchy's condensation test, the given series $\sum f(n)$, that is, $\sum_{n=2}^{\infty} \frac{\log n}{n^p}$ is convergent when $p > 1$ and divergent when $p \leqslant 1$.

17. *Test for convergence* $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log n}$. (R U 1968 H)

Solution. Let the given series be denoted by $\sum f(n)$.

Then $f(n) = \dfrac{1}{\sqrt{n}\,\text{log}n}$.

Obviously $f(n)$ is positive and decreasing as denominator increases with n.

Let a be a positive integer greater than unity.

$$\therefore \ a^n f(a^n) = a^n \frac{1}{\sqrt{(a^n)}\log a^n}.$$

$$= \frac{\sqrt{a^n}}{n\text{log}a} = V_n \text{ (say)}.$$

$$V_{n+1} = \frac{\sqrt{a^{n+1}}}{(n+1)\log a}.$$

$$\therefore \ \frac{V_n}{V_{n+1}} = \frac{n+1}{n} \cdot \frac{\sqrt{a^n}}{\sqrt{a^{n+1}}} = \left(1+\frac{1}{n}\right)\frac{a^{\frac{n}{2}}}{a^{\frac{n+1}{2}}}$$

$$=\left(1+\frac{1}{n}\right)\frac{1}{a^{\frac{1}{2}}}.$$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{V_n}{V_{n+1}}=\frac{1}{\sqrt{a}}, \text{ which is less than 1, as } a>1.$$

Hence, by D' Alembert's ratio test, the series $\Sigma\, V_n$, that is, $\Sigma\, a^n f(a^n)$ is divergent.

Hence, by Cauchy's condensation test, the given series $\Sigma\, f(n)$, that is, $\Sigma\, \dfrac{1}{\sqrt{n}\log n}$ is divergent.

18. *Examine the convergence of the series*—

$$\frac{(\log 2)^2}{2^2}+\frac{(\log 3)^2}{3^2}+\frac{(\log 4)^2}{4^2}+\ldots+\frac{(\log n)^2}{n^2}+\ldots \text{to } \infty.$$

Solution. Let the given series be denoted by $\Sigma\, f(n)$.

Here $\dfrac{(\log 1)^2}{1^2}$, i.e., 0 may be regarded as the first term.

Then $f(n)=\dfrac{(\log n)^2}{n^2}$.

Clearly $f(n)$ is a decreasing function of n as n increases at a faster rate than $\log n$.

Let a be a positive integer greater than unity.

$$\therefore \quad a^n f(a^n)=\frac{a^n(\log a^n)^2}{(a^n)^2}=\frac{a^n(n\log a)^2}{a^{2n}}$$

$$=\frac{a^n.n^2(\log a)^2}{a^{2n}}=\frac{(\log a)^2}{a^n}.\,n^2=V_n \text{ (say)}$$

$$\therefore \quad V_{n+1}=\frac{(\log a)^2}{a^{n+1}}.\,(n+1)^2$$

$$\therefore \quad \frac{V_n}{V_{n+1}}=\frac{n^2}{(n+1)^2}\,a=\frac{1}{\left(1+\frac{1}{n}\right)^2}.\,a$$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{V_n}{V_{n+1}}=a, \text{ which is greater than unity.}$$

Therefore, by D'Alembert's ratio test, ΣV_n, that is, $\Sigma a^n f(a^n)$ is convergent.

Hence, by Cauchy's condensation test, the given series is also convergent.

19. *Test for convergence the following series for $p \in R$—*

$$\frac{1}{(\log 2)^p} + \frac{1}{(\log 3)^p} + \frac{1}{(\log 4)^p} + \ldots + \frac{1}{(\log n)^p} + \ldots \text{to } \infty.$$

(Agra univ 1945)

Solution. Let the given series be denoted by $\Sigma f(n)$.

Then $f(n) = \dfrac{1}{(\log n)^p}$.

If $p > 0$, $f(n)$ is a positive decreasing function of n as $\log n$ increases with n.

Let a be a positive integer, greater than unity.

$$\text{Now} \quad a^n f(a^n) = a^n . \frac{1}{(\log a^n)^p}$$

$$= a^n . \frac{1}{(n \log a)^p} = \frac{a^n}{n^p (\log a)^p} = V_n (\text{say})$$

$$\therefore \quad V_{n+1} = \frac{a^{n+1}}{(n+1)^p (\log a)^p}$$

$$\therefore \quad \frac{V_n}{V_{n+1}} = \frac{a^n}{n^p (\log a)^p} . \frac{(n+1)^p (\log a)^p}{a^{n+1}}$$

$$= \left(1 + \frac{1}{n}\right)^p . \frac{1}{a}.$$

$$\Rightarrow \quad \lim_{n \to \infty} \frac{V_n}{V_{n+1}} = \frac{1}{a}, \text{ which is less than 1 for } a > 1.$$

Therefore, by D' Alembert's ratio test, the series ΣV_n, that is, $\Sigma a^n f(a^n)$ is divergent.

Hence, by Cauchy's condensation test, the given series $\Sigma f(n)$ is also divergent when $p > 0$.

If $p \leqslant 0$, let $p = -m$ where $m \geqslant 0$.

Then $f(n) = \dfrac{1}{(\log n)^p} = (\log n)^m$,

and $\lim\limits_{n\to\infty} f(n) = \lim\limits_{n\to\infty} (\log n)^m = \infty$.

$\therefore \quad \Sigma f(n)$ is divergent when $p \leqslant 0$.

Thus the given series is divergent for all real values of p.

20. *Test the convergence of* $\sum\limits_{n=2}^{\infty} \dfrac{n+2}{n^2-1} \dfrac{1}{(\log n)^{\frac{11}{10}}}$.

(R U 1962 H; Bh U '62 H)

Solution. Let the given series be denoted by $\Sigma f(n)$

Then $f(n) = \dfrac{n+2}{n^2-1} \dfrac{1}{(\log n)^{\frac{11}{10}}}$

Let us compare the given series $\Sigma f(n)$ with the auxiliary series $\Sigma \varphi(n)$, where $\varphi(n) = \dfrac{1}{n} . \dfrac{1}{(\log n)^{\frac{11}{10}}}$.

Now $\dfrac{f(n)}{\varphi(n)} = \dfrac{1+\dfrac{2}{n}}{1-\dfrac{1}{n^2}} \to 1$ as $n \to \infty$ and 1 is finite and non-zero. Hence by comparison test $\Sigma f(n)$ and $\Sigma \varphi(n)$ converge or diverge simultaneously.

Again $\varphi(n) = \dfrac{1}{n} . \dfrac{1}{(\log n)^{\frac{11}{10}}}$ is clearly a positive decreasing function of n for $n (\log n)^{11/10}$ increases with n.

Let a be a positive integer greater than unity.

Then $a^n \varphi(a^n) = a^n . \dfrac{1}{a^n (\log a^n)^{\frac{11}{10}}}$

$$= \frac{1}{(n \log a)^{\frac{11}{10}}}$$

$$= \frac{1}{(\log a)^{\frac{11}{10}}} . \frac{1}{n^{\frac{11}{10}}}$$

$$\Rightarrow \quad \Sigma a^n \varphi(a^n) = \frac{1}{(\log a)^{\frac{11}{10}}} \Sigma \frac{1}{n^{\frac{11}{10}}}.$$

But $\sum \frac{1}{n^{\frac{11}{10}}}$ is convergent as $p = \frac{11}{10} > 1$, which is greater than unity.

Therefore, $\sum a^n \varphi(a^n)$ is convergent.

Hence, by Cauchy's condensation test, the series $\sum \varphi(n)$ is also convergent.

Thus the given series $\sum f(n)$ is also convergent.

21. *Test the convergence of* $\sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}$

(M U 1968 H; B U '62 H)

Solution. Let the given series be denoted by $\sum f(n)$.

Then $f(n) = \frac{1}{(\log n)^{\log n}}$.

As n increases, $(\log n)^{\log n}$ increases. So $f(n)$ is a positive decreasing function of n.

Let a be a positive integer greater than unity.

$$\text{Then } a^n f(a^n) = a^n \frac{1}{(\log a^n)^{\log a^n}}$$

$$= a^n \frac{1}{(n \log a)^{n \log a}}$$

$$= a^n \frac{1}{n^{n \log a}\ (\log a)^{n \log a}}.$$

Let us take $a = 3$ and since $\log a$, that is, $\log 3 > 1$

We get $$\frac{1}{\log 3} < 1$$

and so $$\frac{1}{n^{n \log 3}} < \frac{1}{n^n} \text{ and } \frac{1}{(\log 3)^{n \log 3}} < 1.$$

$$\therefore \quad a^n f(a^n) < \frac{3^n}{n^n} \leqslant \left(\frac{3}{4}\right)^n, \text{ when } n \geqslant 4.$$

But $\sum \left(\frac{3}{4}\right)^n$ is convergent, as it is in G. P. where c. r. is $\frac{3}{4}$.

$\therefore$ $\Sigma a^n f(a^n)$ is convergent. Hence by Cauchy's condensation test, $\Sigma f(n)$ is convergent.

22. *Discuss the convergence of the series*

$$\Sigma \frac{1}{n \log n (\log \log n)^p} \text{ for all values of } p.$$

(P U 1971 H; M U '64 H, '66 H, '70 H, '72 H, '80 H;
B U '59 H, '61 H; B P S C'79)

Solution. Let the given series be denoted by $\Sigma f(n)$.

Then $f(n) = \dfrac{1}{n \log n (\log \log n)^p}$.

Obviously if $p \geqslant 0$, $f(n)$ is *a* positive decreasing function of n.

If $p < 0$, let $p = -m$, $m > 0$. Then $f(n) = \dfrac{(\log \log n)^m}{n \log n}$

which is also a decreasing function of n for the denominator increases at a faster rate than the numerator.

Thus $f(n)$ is a decreasing function of n and also $f(n)$ is positive for all $n \geqslant k$, a suitable integer.

$\therefore$ Cauchy's condensation test is applicable for all p.

Let a be a positive integer greater than unity.

$$\text{Then } a^n f(a^n) = a^n . \frac{1}{a^n \log a^n (\log \log a^n)^p}$$

$$= \frac{1}{n \log a \,[\log (n \log a)]^p}$$

$$= \frac{1}{n \log a \,[\log n + \log \log a]^p}$$

$$= \frac{1}{n \log a.(\log n)^p \left(1 + \dfrac{\log \log a}{\log n}\right)^p}$$

Let us compare the series $\Sigma a^n f(a^n)$ with the series Σ

where $$v_n = \frac{1}{n (\log n)^p}$$

Now $$\frac{a^n f(a^n)}{v_n} = \frac{1}{\log a\left(1+\dfrac{\log\log a}{\log n}\right)^p}$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{a^n f(a^n)}{v_n} = \frac{1}{\log a},$$ which is finite and non-zero.

Hence, by comparison test, the two series $\sum a^n f(a^n)$ and $\sum v_n$ converge or diverge simultaneously.

We know that $\sum \dfrac{1}{n(\log n)^p}$ i.e. $\sum v_n$ is convergent when $p > 1$ and divergent when $p \leqslant 1$.

Hence $\sum a^n f(a^n)$ is convergent when $p > 1$ and divergent when $p \leqslant 1$.

Thus, by Cauchy's condensation test, the given series $\sum f(n)$ in convergent when $p > 1$ and divergent when $p \leqslant 1$.

23. *Determine the convergency or divergency of the following series.*

$$1^p + \left(\frac{1}{2}\right)^p + \left(\frac{1.3}{2.4}\right)^p + \left(\frac{1.3.5}{2.4.6}\right)^p + \cdots \text{ to } \infty.$$

(B P S C 1981; Lucknow Univ '50)

Solution. Neglecting the first term of the given series, as this does not affect the convergence or the divergence of the given series and then denoting the series by $\sum u_n$, we get

$$u_n = \left\{\frac{1.3.5.\ldots.(2n-1)}{2.4.6.\ldots.2n}\right\}^p$$

and $$u_{n+1} = \left\{\frac{1.3.5.\ldots.(2n-1)(2n+1)}{2.4.6.\ldots.2n(2n+2)}\right\}^p$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \left(\frac{2n+2}{2n+1}\right)^p = \left(\frac{2+\dfrac{2}{n}}{2+\dfrac{1}{n}}\right)^p$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{2^p}{2^p} = 1.$$

Hence D' Alembert's ratio test fails.

Now we apply logarithmic ratio test.

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^p}{(2n+1)^p} = \frac{\left(1+\frac{1}{n}\right)^p}{\left(1+\frac{1}{2n}\right)^p}$$

$$\Rightarrow \quad \log\frac{u_n}{u_{n+1}} = p\log\left(1+\frac{1}{n}\right) - p\log\left(1+\frac{1}{2n}\right)$$

$$= p\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\text{to }\infty\right.$$

$$\left. - \frac{1}{2n} + \frac{1}{8n^2} - \frac{1}{24n^3} + \ldots\text{ to }\infty\right]$$

$$\Rightarrow \quad n\log\frac{u_n}{u_{n+1}} = p\left[1 - \frac{1}{2n} + \frac{1}{3n^2} - \ldots\text{ to }\infty\right.$$

$$\left. - \frac{1}{2} + \frac{1}{8n} - \frac{1}{24n^2} + \ldots\text{ to }\infty\right]$$

$$= p\left[\frac{1}{2} - \frac{3}{8n} + \frac{7}{24n^2} - \ldots\text{ to }\infty\right]$$

$$\Rightarrow \quad \lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}\right) = \frac{p}{2}.$$

Hence, by logarithmic ratio test, the given series is convergent or divergent according as

$$\frac{p}{2} > 1 \quad \text{or} \quad \frac{p}{2} < 1$$

$$\Rightarrow \quad p > 2 \quad \text{or} \quad p < 2.$$

If $\frac{p}{2} = 1$ i.e. $p = 2$, this test fails.

Now we apply higher logarithmic ratio test.

$$n\log\frac{u_n}{u_{n+1}} = 1 - \frac{3}{4n} + \frac{7}{12n^2} - \ldots\text{ to }\infty$$

$$\Rightarrow \quad n\log\frac{u_n}{u_{n+1}} - 1 = -\frac{3}{4n} + \frac{7}{12n^2} - \ldots\text{ to }\infty$$

$$\Rightarrow \quad \left[\left(n \log \frac{u_n}{u_{n+1}} - 1\right) \log n\right] = \left(-\frac{3}{4} + \frac{7}{12n} - \ldots\right)\frac{\log n}{n}$$

$$\Rightarrow \quad \lim_{n\to\infty} \left[\left(n \log \frac{u_n}{u_{n+1}} - 1\right) \log n\right] = -\frac{3}{4} \times 0 = 0 < 1.$$

$$\left(\because \quad \lim_{n\to\infty} \frac{\log n}{n} = 0\right)$$

Hence, by higher logarithmic ratio test, the given series is divergent, when $p = 2$.

24. *Show that the series*

$$a^{\frac{1}{m}} + a^{\frac{1}{m}+\frac{1}{m+1}} + a^{\frac{1}{m}+\frac{1}{m+1}+\frac{1}{m+2}} + \ldots \text{to } \infty.$$

is convergent or divergent according as $a < \dfrac{1}{e}$ or $\nless \dfrac{1}{e}$,

(P U 1957 H)

Solution. Let the given series be denoted by $\sum u_n$.

$$\text{Then} \quad u_n = a^{\frac{1}{m}+\frac{1}{m+1}+\frac{1}{m+2}+\ldots+\frac{1}{m+n-1}},$$

$$\text{and} \quad u_{n+1} = a^{\frac{1}{m}+\frac{1}{m+1}+\frac{1}{m+2}+\ldots+\frac{1}{m+n-1}+\frac{1}{m+n}}.$$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{1}{a^{\frac{1}{m+n}}}$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1}{a^0} = \frac{1}{1} = 1.$$

Hence D' Alembert's ratio test fails.

Now we apply logarithmic ratio test.

$$\therefore \quad \log \frac{u_n}{u_{n+1}} = -\frac{1}{m+n} \log a$$

$$\Rightarrow \quad n \log \frac{u_n}{u_{n+1}} = -\frac{n}{m+n} \log a = -\frac{1}{1+\frac{m}{n}} \log a$$

$$\Rightarrow \quad \lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}\right) = -\log a = \log\frac{1}{a}.$$

Hence, by logarithmic ratio test, the given series is convergent or divergent according as

$$\log\frac{1}{a} > 1 \quad \text{or} \quad \log\frac{1}{a} < 1$$

$$\Rightarrow \quad \log\frac{1}{a} > \log e \quad \text{or} \quad \log\frac{1}{a} < \log e$$

$$\Rightarrow \quad \frac{1}{a} > e \quad \text{or} \quad \frac{1}{a} < e$$

$$\Rightarrow \quad a < \frac{1}{e} \quad \text{or} \quad a > \frac{1}{e}.$$

When $a = \frac{1}{e}$, this test fails.

Now we apply higher logarithmic ratio test.

$$n\log\frac{u_n}{u_{n+1}} = -\frac{n}{m+n}\log\frac{1}{e} = \frac{n}{m+n}$$

$$\Rightarrow \quad n\log\frac{u_n}{u_{n+1}} - 1 = \frac{n}{m+n} - 1 = -\frac{m}{m+n}$$

$$\Rightarrow \quad \left\{n\log\frac{u_n}{u_{n+1}} - 1\right\}\log n = -\frac{m\log n}{m+n}$$

$$= -\frac{m}{1+\frac{m}{n}}\left(\frac{\log n}{n}\right)$$

$$\Rightarrow \quad \lim_{n\to\infty}\left[\left\{n\log\frac{u_n}{u_{n+1}} - 1\right\}\log n\right] = -m\times 0 = 0,$$

which is less than 1, $\left(\because \lim_{n\to\infty}\frac{\log n}{n} = 0\right)$.

Hence, by higher logarithmic ratio test the given series is divergent when $a = \frac{1}{e}$.

Thus the given series is convergent or divergent according as $a < \frac{1}{e}$ or $\nless \frac{1}{e}$.

25. *Apply Gauss' Test to examine the convergence of the series* $\frac{1^2}{2^2} + \frac{1^2.3^2}{2^2.4^2} + \frac{1^2.3^2.5^2}{2^2.4^2.6^2} + \ldots$ to ∞. (Rajasthan 1961)

Solution. Let the series be denoted by $\sum u_n$.

Then $u_n = \dfrac{1^2.3^2.5^2 \ldots (2n-1)^2}{2^2.4^2.6^2 \ldots (2n)^2}$

and $u_{n+1} = \dfrac{1^2.3^2.5^2 \ldots (2n-1)^2 . (2n+1)^2}{2^2.4^2.6^2 \ldots (2n)^2 (2n+2)^2}$

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \frac{4n^2\left(1+\frac{1}{n}\right)^2}{4n^2\left(1+\frac{1}{2n}\right)^2}$$

$$= \left(1+\frac{1}{n}\right)^2\left(1+\frac{1}{2n}\right)^{-2}$$

$$= \left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left\{1 - 2.\frac{1}{2n} + 3.\left(\frac{1}{2n}\right)^2 - \ldots\right\}$$

using Binomial theorem for any index

$$= 1 + \frac{1}{n} + \frac{1}{n^2}\left\{-\frac{1}{4} + \text{terms containing } \frac{1}{n^3} \text{ etc}\right\}$$

$$= 1 + \frac{1}{n} + \frac{b_k}{n^2} \text{ where } b_k \to -\frac{1}{4} \text{ as } n \to \infty.$$

$\therefore$ By Gauss' test, the series is divergent, as here $a = 1$.

Alternatively (without applying Gauss' Test)

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \frac{\left(2+\frac{2}{n}\right)^2}{\left(2+\frac{1}{n}\right)^2}$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{4}{4} = 1.$$

Hence D'Alembert's ratio test fails.

Now we apply Raabe's test.

$$\frac{u_n}{u_{n+1}} - 1 = \frac{(2n+2)^2}{(2n+1)^2} - 1 = \frac{4n+3}{(2n+1)^2}.$$

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{4n^2+3n}{4n^2+4n+1} = \frac{4+\frac{3}{n}}{4+\frac{4}{n}+\frac{1}{n^2}}.$$

$$\Rightarrow \lim_{n\to\infty}\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right)\right\} = \frac{4}{4} = 1.$$

Hence Raabe's test fails.

Now we apply De Morgan and Bertrand's test.

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1 = \frac{4n^2+3n}{4n^2+4n+1} - 1 = \frac{-n-1}{4n^2+4n+1}$$

$$\Rightarrow \left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\}\log n = \frac{-n-1}{4n^2+4n+1}.\log n$$

$$= \frac{-n^2-n}{4n^2+4n+1}\cdot\frac{\log n}{n}$$

$$= \frac{-1-\frac{1}{n}}{4+\frac{4}{n}+\frac{1}{n^2}}\cdot\frac{\log n}{n}.$$

$$\Rightarrow \lim_{n\to\infty}\left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\}\log n\right] = 0,$$

which is less than 1.

$$\left(\text{as } \frac{\log n}{n} \to 0, \text{ as } n \to \infty\right)$$

Hence, by De Morgan and Bertrand's test, the given series is divergent.

Problem-Set 5

1. Examine the convergence of the following series—

(i) $\frac{2}{3.4} + \frac{2.4}{3.5.6}x + \frac{2.4.6}{3.5.7.8}x^2 + \ldots$ to ∞, $(x > 0)$

(P U 1951 H)

(ii) $\frac{x}{1}+\frac{1}{2}\cdot\frac{x^2}{3}+\frac{1.3}{2.4}\cdot\frac{x^3}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{x^4}{7}+\ldots\text{to }\infty,\quad (x>0)$

(Agra U 1949; Rajputana '59)

(iii) $1+\frac{3}{7}x+\frac{3.6}{7.10}x^2+\frac{3.6.9}{7.10.13}x^3+\ldots\text{to }\infty,\quad (x>0)$

(M. U. 1985)

2. Discuss the convergence of the following series for all real values of x—

(i) $x^2+\frac{2^2}{3.4}x^4+\frac{2^2.4^2}{3.4.5.6}x^6+\frac{2^2.4^2.6^2}{3.4.5.6.7.8}x^8+\ldots\text{to }\infty.$

(Rajputana 1957)

(ii) $\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\frac{x^7}{7}+\ldots\text{to }\infty.$

(Roorkee Engg 1979)

3. Examine the convergence of the following series for positive values of x—

(i) $1+\frac{2x}{\underline{|2}}+\frac{3^2x^2}{\underline{|3}}+\frac{4^3x^3}{\underline{|4}}+\ldots\text{to }\infty.$

(B P S C 1982; Rajputana U '64; Agra U '62, 64; Roorkee Engg '67)

(ii) $1+\frac{x}{2}+\frac{\underline{|2}}{3^2}x^2+\frac{\underline{|3}}{4^3}x^3+\ldots\text{to }\infty.$

(Agra U 1956; Sagar U '62; B P S C '83)

4. Determine the convergence of the series—

(i) $\frac{1^2}{2^2}+\frac{1^2.3^2}{2^2.4^2}x+\frac{1^2.3^2.5^2}{2^2.4^2.6^2}x^2+\ldots\text{to }\infty,\quad (x>0)$

(ii) $1+\frac{\alpha}{1.\beta}x+\frac{\alpha(\alpha+1)^2}{1.2.\beta(\beta+1)}x^2+\frac{\alpha(\alpha+1)^2(\alpha+2)^2}{1.2.3.\beta(\beta+1)(\beta+2)}x^3+\ldots\text{to }\infty,\quad (x>0)$

5. Examine the convergence of the following series

(i) $\frac{(1+a)(1+b)}{1.2.3}+\frac{(2+a)(2+b)}{2.3.4}+\ldots+\frac{(n+a)(n+b)}{n(n+1)(n+2)}+\ldots\text{to }\infty.$

(ii) $\frac{a}{b}+\frac{a(a+1)}{b(b+1)}+\frac{a(a+1)(a+2)}{b(b+1)(b+2)}+\ldots\text{to }\infty.$

(Agra U 1963)

(iii) $1+\frac{\alpha.\beta}{\gamma.\delta}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)\delta(\delta+1)}$

$+\frac{\alpha(\alpha+1)(\alpha+2).\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)\delta(\delta+1)(\delta+2)}+\ldots\text{to }\infty.$

6. Test the convergence of the following series—

(i) $\sum_{n=2}^{\infty}\frac{1}{n\log n}$. (R U 1969 H)

(ii) $\sum_{n=2}^{\infty}\frac{1}{n(\log n)^3}$. (M U 1965 H)

7. Test for convergence the series—

(i) $\sum_{n=2}^{\infty}\frac{\log n}{n}$.

(ii) $\sum_{n=2}^{\infty}\frac{1}{\log n}$. (B P S C 1982)

(iii) $\sum_{n=2}^{\infty}\frac{\log n}{n^2+1}$.

8. Examine the convergence of the following series—

(i) $x+x^{1+\frac{1}{2}}+x^{1+\frac{1}{2}+\frac{1}{3}}+x^{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}+\ldots\text{to }\infty.$

9. Apply Gauss' test to examine the convergence of the following series—

(i) $1+\frac{2^2}{3^2}+\frac{2^2.4^2}{3^2.5^2}+\frac{2^2.4^2.6^2}{3^2.5^2.7^2}+\ldots\text{to }\infty.$

3.14. Convergence of series of Positive and Negative Terms

In general an infinite series may be of positive as well as negative terms. The tests for convergence of series of positive terms are not applicable for such series.

Alternating Series. An infinite series is called an alternating series if the terms are alternately positive and negative. For example :

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots \text{is an alternating series.}$$

$$1 + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \ldots \text{is not an alternating series.}$$

It is a series of positive and negative terms.

An alternating series is denoted by $\sum\limits_{1}^{\infty} (-1)^{n-1} u_n$

i.e. $u_1 - u_2 + u_3 - u_4 + \ldots$ to ∞.

Theorem 3.24. Leibnitz's Test (or Rule) for Alternating Series.

Statement. *An infinite series, in which the terms are alternately positive and negative, is convergent if each term is numerically less than the preceding term and the terms decrease indefinitely.*

Alternative Statement

If for the alternating series $\sum\limits_{1}^{\infty} (-1)^{n-1} u_n$

(i) $u_{n+1} < u_n$, *for all* n

and (ii) $u_n \to 0$ *as* $n \to \infty$

then the series is convergent. (M U 1978; B. U. '85)

Proof. The given series is

$$u_1 - u_2 + u_3 - u_4 + u_5 - u_6 + \ldots \text{to } \infty.$$

Let S_{2n} denote the sum of the first $2n$ terms of the given series.

Then

$$S_{2n} = u_1 - u_2 + u_3 - u_4 + u_5 - u_6 + \ldots + u_{2n-1} - u_{2n},$$

i.e. $S_{2n} = (u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \ldots + (u_{2n-1} - u_{2n})$.

Since $u_n > u_{n+1}$, that is,

$u_1 > u_2,\ u_3 > u_4,\ u_5 > u_6,\ \ldots,\ u_{2n-1} > u_{2n}$, we get

$u_1 - u_2,\ u_3 - u_4,\ u_5 - u_6,\ \ldots,\ u_{2n-1} - u_{2n}$ are all positive.

$\therefore$ S_{2n} is positive and increases with n. .. (1)

Again, $S_{2n} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \ldots - u_{2n}$.

Clearly $u_2 - u_3$, $u_4 - u_5$, ...are all positive.

So S_{2n} is always less than u_1 (but positive) .. (2)

From (1) and (2) we find that $\{S_{2n}\}$ is monotonic increasing and bounded by 0 and u_1; S_{2n} tends to a limit

$$\leqslant u_1 \text{ and } S_{2n} > 0.$$

Now $S_{2n+1} = S_{2n} + u_{2n+1}$.

$$\therefore \quad \underset{n\to\infty}{\text{Lt}}\, S_{2n+1} = \underset{n\to\infty}{\text{Lt}}\, (S_{2n} + u_{2n+1}) = \underset{n\to\infty}{\text{Lt}}\, S_{2n} + \underset{n\to\infty}{\text{Lt}}\, u_{2n+1}.$$

Since $u_n \to 0$ as $n \to \infty$, therefore $u_{2n+1} \to 0$ as $n \to \infty$.

Hence $\underset{n\to\infty}{\text{Lt}}\, S_{2n+1} = \underset{n\to\infty}{\text{Lt}}\, S_{2n}$.

$\therefore$ S_{2n+1} and S_{2n} tend to the same limit as $n \to \infty$.

Hence S_n tends to a limit as $n \to \infty$.

Thus the given series is convergent.

Note. In an alternating series $u_1 - u_2 + u_3 - u_4 + \ldots$, satisfying the conditions (i) and (ii), the sum of the series may be taken as $u_1 - u_2 + u_3 - u_4 + \ldots + (-1)^{n-1}u_n$ with the error R_n where

$$R_n = (-1)^n\{u_{n+1} - u_{n+2} + u_{n+3} - u_{n+4} + \ldots \text{to } \infty\}.$$

Now $u_{n+1} - u_{n+2} + u_{n+3} - u_{n+4} + \ldots$ is a convergent series by Leibnitz's test and

$$\begin{aligned} |R_n| &= u_{n+1} - u_{n+2} + u_{n+3} - u_{n+4} + \ldots \\ &= u_{n+1} - (u_{n+2} - u_{n+3}) - (u_{n+4} - u_{n+5}) - \\ &< u_{n+1}, \text{ as } \{u_n\} \text{ is monotonic decreasing by condition (i)} \end{aligned}$$

3.15. Absolutely Convergent Series

(Bh U 1964 H, '66 H, '68 H, '69 H, '72 H;
R U '63 H, '66 H, '70 H, '72 H, '74 H;
M U '66 H, '68 H, '75 H, '80 H; B U '60 H;
P U '60 H, '62 H, '64 H, '71 H, '85)

The series $\sum u_n$, containing positive and negative terms, is said to be absolutely convergent if the series $\sum |u_n|$ is convergent.

$\sum |u_n|$ denotes the series of absolute values of the terms of $\sum u_n$. If $\sum u_n$ is itself a series of positive terms then

$$\sum |u_n| = \sum u_n.$$

Examples. (i) The series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$ is absolutely convergent, because the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ is convergent, being a G.P. with common ratio $= \frac{1}{2} < 1$.

(ii) The series $1 - \frac{1}{2^3} + \frac{1}{3^3} - \ldots$ is absolutely convergent, as the series $1 + \frac{1}{2^3} + \frac{1}{3^3} + \ldots$ is convergent.

3.16. Conditionally Convergent Series or Semi-Convergent Series

(M U 1975 H; Bh U '66 H, '68 H, '70 H, '72 H; R U '66 H; P U '63 H, '72 H)

The series $\sum u_n$, containing positive and negative terms, is said to be conditionally convergent or semi-convergent or non-absolutely convergent if $\sum u_n$ is convergent but $\sum | u_n |$ is divergent.

Examples. (i) The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ is conditionally convergent as it is convergent, by Leibnitz's Test for alternating series and $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is divergent because $\sum \frac{1}{n^p}$ is divergent when $p \leqslant 1$.

(ii) The series $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots$ is conditionally convergent as it is convergent, by Leibnitz's Test for alternating series, and $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots$ is divergent, because $\sum \frac{1}{n^p}$ is divergent when $p \leqslant 1$.

Theorem 3.25. An Absolutely Convergent Series is Convergent.

(Bh U 1967 H, '69 H, '76; M U '66 H '68 H, '75 H; B U '54 H, '58 H, '60 H, '68 H; P U '56 H, '58 H, '60 H, '62 H, '66 H, '68 H, '73 H, '85)

Proof. Let the series $\sum u_n$ be absolutely convergent; then by definition, the series $\sum | u_n |$ is convergent.

We have $u_n + | u_n | = \begin{cases} 2u_n, \text{ when } u_n > 0; \\ 0 \quad , \text{ when } u_n < 0. \end{cases}$

Therefore every term of the series $\Sigma (u_n + | u_n |) \geqslant 0$.

Now $\Sigma (u_n + | u_n |)$ as well as $\Sigma 2 | u_n |$ are series of positive terms for which $u_n + | u_n | \leqslant 2 | u_n |$.

But $\Sigma\, 2 | u_n |$ is convergent. So by comparison test

$\Sigma\, (u_n + | u_n |$ is also convergent.

Hence $\Sigma\, [(u_n + | u_n |) - | u_n |]$, i.e., Σu_n is convergent.

Note. The converse of the above theorem is not necessarily true. (P U 1964 H, '70 H, '73 H; B U '63 H; Bh U '66 H)

For example, consider the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$

This series is convergent, by Leibnitz's Test for alternating series; but the series of absolute values of the terms of the above series i.e. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ is divergent.

Hence the given series is not absolutely convergent though it is convergent.

Theorem 3.26. *The terms of an absolutely convergent series can be rearranged without affecting the convergence and the sum of the series.* (B P S C 1979, '81; P U '59 H, '69 H, '71 H; M U '65 H, '71 H, '73 H, '80 H; Bh U '68 H, '72 H; R U '67 H, '70 H, '72 H)

Proof. Let the absolutely convergent series be Σu_n. Then, by definition of absolutely convergence, the series $\Sigma\, | u_n |$ is convergent. By the rearrangement of the terms of Σu_n we get a new series, say Σv_n so that every v is a u and every u is a v.

We have $u_n + | u_n | = 2u_n$ or 0, according as $u_n >$ or < 0.

Since the terms of $\Sigma (u_n + | u_n |) \leqslant$ the corresponding terms of $\Sigma 2 | u_n |$ and $\Sigma\, | u_n |$ is a convergent series of positive terms, by comparison test $\Sigma (u_n + | u_n |)$ is also a convergent series of positive terms.

Let $\Sigma\, | u_n | = S$ and $\Sigma (u_n + | u_n |) = s$. then

$$\Sigma u_n = s - S.$$

Since $\Sigma\, | u_n |$ and $\Sigma (u_n + | u_n |)$ are series of positive terms, their sums are unaltered by any re-arrangement of terms.

$\therefore \quad \sum |v_n| = S,\ \sum(v_n + |v_n|) = s$ and

$$\sum v_n = s - S = \sum u_n.$$

Hence the theorem.

Note. The above property of rearrangement of terms does not hold in general for series of positive and negative terms if the series is not absolutely convergent. This is clear from the following example—

Take the infinite series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$ .. (I)

This is not absolutely convergent, for $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots.$ is divergent.

Now consider the infinite series

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \ldots. \quad .. \quad \text{(II)}$$

which has been obtained by the re-arrangement of the terms of the series (I) under consideration.

Evidently the series (I) is semi-convergent.

Let s be its sum.

Let $s_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$to n terms,

$\sigma_n = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \ldots.$to n terms,

then $\sigma_{3n} = (1 - \frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{6} - \frac{1}{8}) + (\frac{1}{5} - \frac{1}{10} - \frac{1}{12})$ $+\ldots.$to n brackets

$\Rightarrow \quad \sigma_{3n} = (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{6} - \frac{1}{8}) + (\frac{1}{10} - \frac{1}{12})$ $+\ldots.$to n brackets

$\Rightarrow \quad \sigma_{3n} = \frac{1}{2}(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$to $2n$ terms$)$

$\Rightarrow \quad \sigma_{3n} = \frac{1}{2} s_{2n}$

$$\therefore \quad \lim_{n\to\infty} \sigma_{3n} = \frac{1}{2} \lim_{n\to\infty} s_{2n} = \frac{1}{2}s.$$

But $\lim\limits_{n\to\infty} \sigma_{3n+1} = \lim\limits_{n\to\infty} \sigma_{3n+2} = \lim\limits_{n\to\infty} \sigma_{3n}.$

$$\therefore \quad \lim_{n\to\infty} \sigma_n = \frac{1}{2}s.$$

Hence the second series (II) converges and its sum is $\frac{1}{2}s$.

The above example shows that a re-arrangement of the terms of a non-absolutely convergent series may alter its sum.

Theorem 3.27. *In an absolutely convergent series, the series formed by its positive terms alone is convergent and the series formed by its negative terms alone is convergent.*

(M U 1969 H; Bh U '74 H; B P S C '80)

Proof. Let the absolutely convergent series be $\sum u_n$ and suppose that

$$S_n = \sum_{n=1}^{n} u_n = u_1 + u_2 + u_3 + \ldots + u_n,$$

$$\text{and } s_n = \sum_{n=1}^{n} | u_n | = | u_1 | + | u_2 | + | u_3 | + \ldots + | u_n | .$$

If the sums of the positive and negative terms separately in S_n be denoted by P and $-N$ respectively, then

$$S_n = P - N \text{ and } s_n = P + N.$$

$$\therefore \quad 2P = S_n + s_n \text{ and } 2N = s_n - S_n.$$

Since the series $\sum u_n$ is absolutely convergent, therefore S_n and s_n both tend to finite limit as $n \to \infty$.

Let these limits be S and T respectively.

$$\therefore \quad P \to \tfrac{1}{2}(S + T) \text{ and } N \to \tfrac{1}{2}(T - S), \text{ as } n \to \infty.$$

Hence the theorem.

Theorem 3.28. *To prove that in a conditionally convergent series $\sum u_n$, the positive as well as the negative terms form two divergent series separately, the terms of either being taken in the order in which they occur.* (R U 1971 H, '73 H; Bh U '74 H)

Proof. Let $S_n = \sum_{n=1}^{n} u_n = u_1 + u_2 + u_3 + \ldots + u_n,$

$$\text{and } s_n = \sum_{n=1}^{n} | u_n |$$

$$= | u_1 | + | u_2 | + | u_3 | + \ldots + | u_n | .$$

If the sums of the positive and negative terms separately in S_n be denoted by P and $-N$ respectively, then

$$S_n = P - N, \quad \ldots \quad (1)$$

and

$$s_n = P + N. \quad \ldots \quad (2)$$

$$\therefore \qquad 2P = S_n + s_n, \qquad .. \quad (3)$$

and $$2N = s_n - S_n. \qquad .. \quad (4)$$

Since $\sum u_n$ is conditionally convergent, $\sum_{n=1}^{n} u_n \to$ a finite quantity and $\sum_{n=1}^{n} |u_n| \to \infty$, as $n \to \infty$.

$$\therefore \qquad S_n \to \text{a finite limit, say } l,$$

and $$s_n \to \text{infinity, as } n \to \infty.$$

Hence P and N both tend to infinity separately.

Thus the positive as well as the negative terms of a conditionally convergent series form two divergent series separately.

3.17. Convergence of Some Useful Series

(A) Binomial Series.

To discuss the convergence of

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots \text{ to } \infty.$$

Solution. (Elementary discussion) Let the given series be considered after the first term and be denoted by $\sum u_n$.

Then $u_r = \dfrac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r$

and $u_{r+1} = \dfrac{n(n-1)(n-2)\dots(n-r+1)(n-r)}{(r+1)!}x^{r+1}.$

$$\therefore \quad \frac{u_r}{u_{r+1}} = \frac{(r+1)!}{r!} \; \frac{1}{n-r} \cdot \frac{x^r}{x^{r+1}}$$

$$= \frac{(r+1)}{n-r} \cdot \frac{1}{x}$$

$$= \frac{1+\dfrac{1}{r}}{\dfrac{n}{r}-1} \cdot \frac{1}{x}$$

$$\Rightarrow \quad \lim_{r\to\infty} \frac{u_r}{u_{r+1}} = -\frac{1}{x}$$

$$\Rightarrow \quad \lim_{r\to\infty} \left| \frac{u_r}{u_{r+1}} \right| = \left| \frac{1}{x} \right|.$$

If $\left| \frac{1}{x} \right| > 1 \Rightarrow |x| < 1$, then the series $\Sigma\, |u_n|$ is convergent. So Σu_n is absolutely convergent and hence convergent when $|x| < 1$ i.e. $-1 < x < 1$.

Rigorous Discussion.

As in the elementary discussion we can prove that the binomial series is absolutely convergent when $|x| < 1$

i.e. $-1 < x < 1$. When $|x| = 1$ we get $x = 1, -1$.

(a) When $x = 1$, $\frac{u_r}{u_{r+1}} = \frac{r+1}{n-r}$.

If $r > n$ then $\frac{u_r}{u_{r+1}} < 0$. So the series is alternating after some stage, consecutive terms having opposite signs.

$$\text{Now} \left| \frac{u_r}{u_{r+1}} \right| = \frac{r+1}{r-n} = \frac{(r-n)+(n+1)}{r-n}$$

$$= 1 + \frac{n+1}{r-n} = 1 + \frac{a_r}{r} \text{ where } a_r = \frac{n+1}{1-\frac{n}{r}}.$$

The following cases arise—

(i) If $n > -1$ then $\lim_{r\to\infty} a_r = \lim_{r\to\infty} \frac{n+1}{1-\frac{n}{r}} = n+1 > 0$.

By definition of limit, given $\epsilon > 0$ there is *a* positive integer *m* such that

$$|a_r - (n+1)| < \epsilon \quad \text{for} \quad r \geq m;$$

$$\therefore \quad a_r > (n+1) - \epsilon \quad \text{for} \quad r \geq m.$$

$\therefore\ a_r > k$ for $r \geq m$ where $n+1 > k = (n+1) - \epsilon > 0$.

Now $\left|\dfrac{u_m}{u_r}\right| = \left|\dfrac{u_m}{u_{m+1}}\right| \left|\dfrac{u_{m+1}}{u_{m+2}}\right| \cdots \left|\dfrac{u_{r-1}}{u_r}\right|$

$$= \left(1 + \frac{a_m}{m}\right)\left(1 + \frac{a_{m+1}}{m+1}\right)\cdots\left(1 + \frac{a_{r-1}}{r-1}\right)$$

$$> \left(1 + \frac{k}{m}\right)\left(1 + \frac{k}{m+1}\right)\cdots\left(1 + \frac{k}{r-1}\right)$$

$$> 1 + k\left(\frac{1}{m} + \frac{1}{m+1} + \dots + \frac{1}{r-1}\right) \to \infty \text{ as } r \to \infty.$$

$\therefore \quad \dfrac{|u_m|}{|u_r|} \to \infty$ i.e. $|u_r| \to 0$ as $r \to \infty$

because $|u_m|$ is a fixed number.

Also $\left|\dfrac{u_r}{u_{r+1}}\right| = 1 + \dfrac{n+1}{r-n} > 1$ for $r > n$.

So $|u_r| > |u_{r+1}|$ after some stage.

Hence by alternating series test, the binomial series is convergent,

(ii) If $n = -1$, the series becomes $1 - 1 + 1 - 1 + \dots$. which oscillates finitely.

(iii) If $n < -1$, then $n + 1 < 0$.

Let $n + 1 = -m$ where $m > 0$.

Then $|u_r| = \left|\dfrac{n(n-1)(n-2)\dots(n-r+1)}{\lfloor r}\right|$

$$= \left|(-1)^r \frac{(m+1)(m+2)(m+3)\dots(m+r)}{1.2.3\dots r}\right|$$

$$= (1+m)\left(1 + \frac{m}{2}\right)\left(1 + \frac{m}{3}\right)\cdots\left(1 + \frac{m}{r}\right)$$

$$> 1 + m\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{r}\right) \to \infty \text{ as } r \to \infty.$$

$\therefore \quad |u_r| \to \infty$ as $r \to \infty$. Hence the series oscillates infinitely.

(b) When $x = -1, \dfrac{u_r}{u_{r+1}} = \dfrac{r+1}{n-r} \cdot \dfrac{1}{-1} = \dfrac{r+1}{r-n}$.

$\therefore$ For $r > n$, the terms u_r and u_{r+1} have the same sign. So the terms, from some stage onwards, are of the same sign.

$$\text{Now } \frac{u_r}{u_{r+1}} = \frac{r-n+(1+n)}{r-n}$$

$$= 1 + \frac{n+1}{r-n}$$

$$= 1 + \frac{n+1}{r} \cdot \frac{1}{1-\dfrac{n}{r}}$$

$$= 1 + \frac{n+1}{r} \cdot \frac{1-\dfrac{n}{r}+\dfrac{n}{r}}{1-\dfrac{n}{r}}$$

$$= 1 + \frac{n+1}{r}\left\{1 + \frac{n}{r\left(1-\dfrac{n}{r}\right)}\right\}$$

$$= 1 + \frac{n+1}{r} + \frac{n(n+1)}{r^2} \cdot \frac{1}{1-\dfrac{n}{r}}$$

$$= 1 + \frac{n+1}{r} + \frac{b_r}{r^2}$$

$$\text{where } \quad b_r = \frac{n(n+1)}{1-\dfrac{n}{r}} \to n(n+1), \text{ as } r \to \infty.$$

$\therefore$ By Gauss' test on convergence, the series converges or diverges according as $n+1 > 1$ or < 1 i.e. $n > 0$ or < 0.

(c) When $|x| > 1, \quad \left|\dfrac{u_r}{u_{r+1}}\right| = \left|\dfrac{r+1}{n-r} \cdot \dfrac{1}{x}\right|$

$$= \left| \frac{1 + \frac{1}{r}}{\frac{n}{r} - 1} \right| \cdot \frac{1}{|x|} < 1 \text{ for all sufficiently large values of } r.$$

So $| u_r | < | u_{r+1} |$ i.e. the numerical values of the terms go on increasing as r increases. So the series is divergent or oscillates infinitely.

(B) Exponential Series.

To discuss the convergence of the exponential series, viz.,

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \text{ to } \infty.$$

Solution. Let the given series be $\sum u_n$.

Then $u_n = \dfrac{x^{n-1}}{(n-1)!}$

and $u_{n+1} = \dfrac{x^n}{n!}$.

$$\therefore \quad \left| \frac{u_n}{u_{n+1}} \right| = \left| \frac{x^{n-1}}{x^n} \cdot \frac{n!}{(n-1)!} \right|$$

$$= \left| \frac{1}{x} \right| \cdot \left| \frac{(n-1)!\,n}{(n-1)!} \right|$$

$$= \left| \frac{n}{x} \right|$$

$\Rightarrow \lim_{n \to \infty} \left| \dfrac{u_n}{u_{n+1}} \right| = \infty$, which is greater than unity for all finite values of x.

Hence, by D' Alembert's ratio test, the series $\Sigma \mid u_n \mid$ is convergent for all finite values of x

So $\sum u_n$ is absolutely convergent and hence convergent.

(C) Logarithmic Series.

(i) *To discuss the convergence of the series*

$$1 + \frac{x}{1!} \log_e a + \frac{x^2}{2!} (\log_e a)^2 + \ldots \textit{ to } \infty.$$

Solution. Let the given series be $\sum u_n$.

Then $u_n = \frac{x^n}{n!}(\log_e a)^n$

and $u_{n+1} = \frac{x^{n+1}}{(n+1)!}(\log_e a)^{n+1}$.

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{x^n}{x^{n+1}} \cdot \frac{(n+1)!}{n!} \cdot \frac{(\log_e a)^n}{(\log_e a)^{n+1}}.$$

$$\therefore \quad \left| \frac{u_n}{u_{n+1}} \right| = \left| \frac{1}{x} \right| \cdot \left| (n+1) \cdot \frac{1}{\log_e a} \right|$$

$$\Rightarrow \lim_{n \to \infty} \left| \frac{u_n}{u_{n+1}} \right| = \infty, \text{ for all finite values of } x.$$

Hence, by D' Alembert's ratio test, the series $\sum |u_n|$ is convergent for all finite values of x. So $\sum u_n$ is absolutely convergent and hence convergent for all finite real values of x.

(ii) *To discuss the convergence of the logarithmic series, viz.*

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \textit{ to } \infty.$$

Solution. Let the given series be $\sum u_n$.

Then $u_n = (-1)^{n-1} \frac{x^n}{n}$

and $u_{n+1} = (-1)^n \frac{x^{n+1}}{n+1}$.

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(-1)^{n-1}}{(-1)^n} \cdot \frac{x^n}{x^{n+1}} \cdot \frac{n+1}{n}$$

$$= -\frac{1}{x}\left(1 + \frac{1}{n}\right)$$

$$\Rightarrow \lim_{n \to \infty} \left| \frac{u_n}{u_{n+1}} \right| = \left| \frac{1}{x} \right|.$$

When $\left| \frac{1}{x} \right| > 1$ i.e. $|x| < 1$, the series $\sum |u_n|$ is convergent.

So the given series $\sum u_n$ is absolutely convergent and hence convergent if $|x| < 1$.

When $\left|\frac{1}{x}\right| < 1$ i.e. $|x| > 1$, the series $\sum u_n$ is divergent.

So $\sum u_n$ is not absolutely convergent when $|x| > 1$.

When $|x| = 1$, then the following cases are noteworthy :

CASE 1. When $x = 1$, then the given series becomes

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots \text{ to } \infty.$$

This is an alternating series. Each term of this series is numerically less than the preceding term and $\lim\limits_{n\to\infty} \frac{1}{n} = 0$.

Hence, by Leibnitz's rule, the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ is convergent,

CASE 2. When $x = -1$, the given series becomes

$$-\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots \text{ to } \infty\right),$$

which is obviously divergent to $-\infty$.

3.18. Working Rule for Testing the Convergence of Arbitrary Series

By observation check whether the series consists of positive as well as negative terms.

(i) If so, find $\lim\limits_{n\to\infty} u_n$ if possible. If the limit is not zero the series is divergent.

(ii) If $\lim\limits_{n\to\infty} u_n = 0$ apply Leibnitz's test if the series is alternating series for which $\{u_n\}$ is monotonic decreasing.

(iii) If Leibnitz's test is not applicable, test the series for absolute convergence.

WORKED OUT EXAMPLES

1. *Test the convergency of the series*

$$2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \ldots \textit{to } \infty.$$

[Rajputana 1959; Agra 1944; Punjab 1941]

Solution. Here $u_n = (-1)^{n-1} \cdot \dfrac{n+1}{n} = (-1)^{n-1}\left(1 + \dfrac{1}{n}\right)$.

So $u_n \to \pm 1 \neq 0$.

But the necessary condition for a series to be convergent is $u_n \to 0$.

So, the given series is not convergent.

2. *Test the convergence of the series*

$$\frac{1}{x} - \frac{1}{x+a} + \frac{1}{x+2a} - \frac{1}{x+3a} + \ldots \text{to } \infty,$$

x and a being positive. [Bh U 1963; P U '46]

Solution. Since x and a are positive, therefore the terms of the series are alternately positive and negative. Each term is numerically less than the preceding term,

$$\text{as } x + (n-1)\,a < x + na, \text{ for all } n.$$

$$\therefore \quad \frac{1}{x+(n-1)a} > \frac{1}{x+na}$$

i.e. $u_n > u_{n+1}$ for all n.

Also, numerically $u_n = \dfrac{1}{x+(n-1)a} \to 0$ as $n \to \infty$.

So, by Leibnitz's theorem on alternating series, the given series is convergent.

3. *Test for convergence the series, whose nth term is*

$$\frac{(-1)^n}{(x+n)(y+n)},\ (x, y > 0).$$ [P U 1966]

Solution. Here $|\,u_n\,| = \dfrac{1}{(x+n)(y+n)}$

and $$|\,u_{n+1}\,| = \frac{1}{(x+n+1)(y+n+1)}.$$

$\therefore \quad |\,u_{n+1}\,| < |\,u_n\,|$, for all n.

Also $\underset{n\to\infty}{Lt}\ u_n = 0$.

The given series is an alternating series with monotonic decreasing terms in absolute values.

Hence, by Leibnitz's test, the given series i.e., $\Sigma\, u_n$ is convergent.

4. *Examine for convergence and divergence the series*

$$\tfrac{2}{1} - \tfrac{3}{2}x + \tfrac{4}{3}x^2 - \dots + (-1)^{n-1}.\left(\frac{n+1}{n}\right)x^{n-1} + \dots$$

(P U 1960)

Solution. If $x > 0$ it is an alternating series whose nth term

$$u_n = \left(\frac{n+1}{n}\right)x^{n-1}$$

$$\therefore \quad \frac{u_n}{u_{n-1}} = \frac{\dfrac{n+1}{n}x^{n-1}}{\dfrac{n}{n-1}x^{n-2}} = \frac{n^2-1}{n^2}x.$$

The series is m.d. if $u_n \leqslant u_{n-1}$ i.e., $\dfrac{u_n}{u_{n-1}} \leqslant 1$.

$\therefore$ It is m.d. if $\dfrac{n^2-1}{n^2}x \leqslant 1$ which holds good for all $x \leqslant 1$.

Also if $x < 1$, $u_n = \dfrac{n+1}{n}x^{n-1} \to 0$ as $n \to \infty$.

$\therefore$ If $x < 1$, the series is convergent.

If x is negative, let $x = -y$ where y is positive. Then the series becomes $\tfrac{2}{1} + \tfrac{3}{2}y + \tfrac{4}{3}y^2 + \dots + \dfrac{n+1}{n}y^{n-1} + \dots$

$$\text{Then, } \frac{u_n}{u_{n+1}} = \frac{n+1}{n}y^{n-1} \Big/ \frac{n+2}{n+1}y^n = \frac{(n+1)^2}{n(n+2)}\cdot\frac{1}{y}$$

$$= \frac{\left(1+\dfrac{1}{n}\right)^2}{1+\dfrac{2}{n}}\cdot\frac{1}{y}.$$

$$\therefore \quad \underset{n\to\infty}{Lt}\ \frac{u_n}{u_{n+1}} = \frac{1}{y}.$$

If $y < 1$ i.e. $x > -1$, $\underset{n\to\infty}{Lt}\dfrac{u_n}{u_{n+1}} > 1$ and the series is convergent.

If $y > 1$ i.e. $x < -1$, the series is divergent.

If $y = 1$, $u_n = \dfrac{n+1}{n} > \dfrac{1}{n}$ and so $\Sigma\, u_n$ is divergent.

$\therefore$ The series is convergent only when $|x| < 1$.

5. *Examine the convergence of the series*—

$$\sum_1^{\infty} \frac{x^n}{1+n^2}.$$

Solution. The series is a series of positive terms if $x > 0$ and it is an alternating series when $x < 0$.

Here $u_n = \dfrac{x^n}{1+n^2}$ and $u_{n+1} = \dfrac{x^{n+1}}{1+(n+1)^2}$.

$$\therefore \quad \frac{|u_n|}{|u_{n+1}|} = \frac{1+(n+1)^2}{1+n^2} \times \left|\frac{1}{x}\right|$$

$$= \frac{n^2+2n+2}{n^2+1} \times \frac{1}{|x|}$$

$$= \frac{n^2\left(1+\frac{2}{n}+\frac{2}{n^2}\right)}{n^2\left(1+\frac{1}{n^2}\right)} \cdot \frac{1}{|x|}$$

$$= \frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}} \cdot \frac{1}{|x|}.$$

$$\therefore \quad \lim_{n\to\infty} \frac{|u_n|}{|u_{n+1}|} = \frac{1}{|x|}.$$

If $\dfrac{1}{|x|} > 1$ i.e. $|x| < 1$, then $\Sigma\, |u_n|$ is convergent.

Hence $\Sigma\, u_n$ is absolutely convergent and hence convergent when $|x| < 1$ i.e. $-1 < x < 1$.

If $x = 1$, the series becomes

$$\frac{1}{1+1^2} + \frac{1}{1+2^2} + \frac{1}{1+3^2} + \dots$$

For it $u_n = \dfrac{1}{1+n^2}$; take $v_n = \dfrac{1}{n^2}$.

$$\therefore \quad \lim_{n \to \infty} \frac{u_n}{v_n} = \lim_{n \to \infty} \frac{n^2}{1+n^2} = 1.$$

But $\Sigma v_n = \Sigma \dfrac{1}{n^2}$ is convergent as $p = 2 > 1$ in $\Sigma \dfrac{1}{n^p}$.

$\therefore$ Σu_n is convergent when $x = 1$.

If $x = -1$, the series becomes

$$-\frac{1}{1+1^2} + \frac{1}{1+2^2} - \frac{1}{1+3^2} + \dots$$

which is alternating and for it the absolute values of the terms are monotonic decreasing and the absolute value of nth term $= \dfrac{1}{1+n^2} \to 0$ as $n \to \infty$.

$\therefore$ By Leibnitz's test, the series is convergent.

If $x > 1$, it is a series of positive terms and as before

$$\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x} < 1.$$

$\therefore$ The series is divergent.

If $x < -1$, let $y = -x$; then $y > 1$ and the series becomes $-\dfrac{y}{1+1^2} + \dfrac{y^2}{1+2^2} - \dfrac{y^3}{1+3^2} + \dots$ which is alternating.

For this series $|u_n| = \dfrac{y^n}{1+n^2}$.

$$\therefore \quad 2|u_n| = \frac{2y^n}{1+n^2} > \frac{2y^n}{n^2+n^2} = \frac{y^n}{n^2}.$$

Now $n \log y > 2 \log n$, after some stage as $y > 1$.

$\therefore$ $\log y^n > \log n^2$ and hence $y^n > n^2$ after some stage.

$\therefore$ The absolute value of u_n does not tend to zero as

$$n \to \infty.$$

Hence the series is not convergent.

Problem-Set 6

1. Show that the infinite series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots.$ is divergent but the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$ is convergent. (M U 1978; P U '66)

2. Examine the series for convergence—

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \ldots. \qquad \text{(P U 1963)}$$

3. Prove that the series

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots.\text{to } \infty$$

is conditionally convergent. (P U 1978, '67)

4. Determine whether the following series is convergent or divergent—

$$\frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+3} - \ldots. \qquad \text{(Mith. U. 1981)}$$

5. Examine the convergence of the following series—

(i) $1 - \dfrac{a}{\lfloor 1} + \dfrac{a^2}{\lfloor 2} - \dfrac{a^3}{\lfloor 3} + \ldots.$ where $0 < a < 1$.

(ii) $\dfrac{a}{1+a} - \dfrac{a^2}{1+a^2} + \dfrac{a^3}{1+a^3} - \ldots.$ where $0 < a < 1$. (P U 1954)

(iii) $\frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \ldots\ldots$ to ∞. (B. U. 1971)

6. If $a_n = \left(1 + \dfrac{x}{n}\right)e^{-x/n}$, show that the series $\Sigma\, v_n$, where $v_n = a_n - 1$, is absolutely convergent for all values of x.

7. Examine the convergence of the series for $x \in R$—

(i) $\dfrac{x}{1.2} + \dfrac{x^2}{3.4} + \dfrac{x^3}{5.6} + \dfrac{x^4}{7.8} + \ldots.$

(ii) $1 + \dfrac{x}{3} + \dfrac{x^2}{5} + \dfrac{x^3}{7} + \ldots.$

(iii) $\sum\limits_{1}^{\infty} n^{-k} \cos nx$. **(B P S C 1979)**

8. Test the convergence of the following series—

(i) $\sum\limits_{1}^{\infty} (-1)^n \dfrac{x^n}{\sqrt{n}}$. **(B P S C 1978)**

(ii) $\sum\limits_{1}^{\infty} (-1)^n \dfrac{\log n}{\sqrt{n}}$. **(B P S C 1982)**

9. Discuss the convergence of the series—

$$1 - \frac{(x-1)}{\lfloor 1} + \frac{(x-1)(x-2)}{\lfloor 2} - \frac{(x-1)(x-2)(x-3)}{\lfloor 3} + \dots$$

for all real values of x. **(B P S C 1980)**

□□□

CHAPTER 4

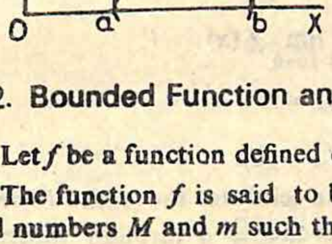
CONTINUITY AND DIFFERENTIABILITY

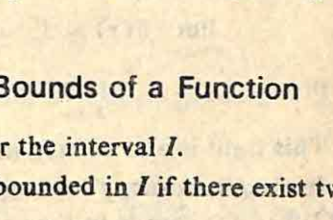
4.1. Real Function of One Variable

Let I be an interval in R. If f is a rule corresponding to which each $x \in I$ is associated with unique element $y \in R$ then f is called a real function of one variable x. The interval I is called the domain of definition of the function f. This function is written as $y = f(x)$, $x \in I$ or simply $f(x)$, $x \in I$.

If I is an open interval (a, b) i.e. $a < x < b$ then the function f is defined over an open interval and if I is a closed interval $[a, b]$ i.e. $a \leqslant x \leqslant b$ then the function f is defined over a closed interval.

The graph of f defined over the open interval (a, b) and closed interval $[a, b]$ are shown as below—





4.2. Bounded Function and Bounds of a Function

Let f be a function defined over the interval I.

The function f is said to be bounded in I if there exist two real numbers M and m such that

$$m \leqslant f(x) \leqslant M \;\forall\; x \in I.$$

M and m are called an upper bound and a lower bound respectively. By continuum property of R, we know it has the least upper bound and the greatest lower bound.

4.3. Limit of a Function

(i) Limit from the right (or Right-hand limit).

A function $f(x)$ is said to tend to the limit l as x tends to a from the right if given $\epsilon > 0$ there exists a number $\delta > 0$ such that $|f(x) - l| < \epsilon$ whenever $a < x < a + \delta$.

This number l is called the right-hand limit of $f(x)$ at $x = a$ and it is denoted by

$$\lim_{x \to a+} f(x) = l \text{ or } \lim_{x \to a+0} f(x) = l$$

or
$$\lim_{h \to 0} f(a + h) = l.$$

This limit is also written as $f(a + 0)$.

In simple words, $f(x)$ is said to tend to the limit l from the right if $f(x)$ tends to l as x approaches a through values of x greater than a.

Working Rule for finding the limit from the right at $x = a$:

(α) Put $a + h$ for x in $f(x)$ to get $f(a + h)$.

(β) Make $h \to 0$ in $f(a + h)$.

(ii) Limit from the left (or Left hand limit).

A function $f(x)$ is said to tend to the limit l' as x tends to a from the left if given $\epsilon > 0$ there exists a number $\delta > 0$ such that $|f(x) - l'| < \epsilon$ whenever $a - \delta < x < a$.

This number l' is called the left-hand limit of $f(x)$ at $x = a$ and it is denoted by

$$\lim_{x \to a-} f(x) = l' \text{ or } \lim_{x \to a-0} f(x) = l'$$

or
$$\lim_{h \to 0} f(a - h) = l'.$$

This limit is also written as $f(a - 0)$.

In simple words, $f(x)$ is said to tend to the limit l' from the left if $f(x)$ tends to l' as x approaches a through values of x smaller than a.

Working Rule for finding the limit from the left at $x = a$:

(A) Put $a - h$ for x in $f(x)$ to get $f(a - h)$.

(B) Make $h \to 0$ in $f(a - h)$.

(iii) Limit of $f(x)$ at $x = a$.

A function $f(x)$ is said to tend to the limit l at $x = a$ if given $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$|f(x) - l| < \epsilon \text{ whenever } |x - a| < \delta$$

i.e. $l - \epsilon < f(x) < l + \epsilon$ whenever $a - \delta < x < a + \delta$.

In simple words, $f(x)$ is said to tend to the limit l at $x = a$ if $f(x)$ tends to l as x approaches a through values of x greater than a as well as values of x smaller than a.

The limit l of $f(x)$ at $x = a$ is denoted by $\lim_{x \to a} f(x) = l$.

It is clear from above that $f(x)$ tends to the limit l at $x = a$ implies

$$\lim_{x \to a+0} f(x) = \lim_{x \to a-0} f(x) = \lim_{x \to a} f(x) = l$$

provided both the right-hand and left-hand limits exist for $x = a$. If the domain of definition of $f(x)$ is $[a, b]$ then existence of $\lim_{x \to a-0} f(x)$ does not arise; similarly the question of existence of $\lim_{x \to b+0} f(x)$ also does not arise.

Example. Let $f(x)$ be a real function defined as follows—

$$f(x) = \begin{cases} 2 - x, & x \leqslant 1 \\ 3x, & x > 1. \end{cases}$$

Here $\lim_{x \to 1-0} f(x) = \lim_{h \to 0} f(1 - h)$

$$= \lim_{h \to 0} \{2 - (1 - h\} = 1,$$

[using $f(x) = 2 - x$]

$$\lim_{x \to 1+0} f(x) = \lim_{h \to 0} f(1 + h)$$

$$= \lim_{h \to 0} 3(1 + h) = 3. \quad [\text{using } f(x) = 3x]$$

As $\lim_{x \to 1-0} f(x) \neq \lim_{x \to 1+0} f(x)$, we get

$$\lim_{x \to 1} f(x) \text{ does not exist.}$$

Next consider the function

$$\varphi(x) = \begin{cases} 2 - x, & x \leqslant 1 \\ x, & x > 1. \end{cases}$$

As before $\lim\limits_{x \to 1-0} \varphi(x) = 1$,

$$\lim_{x \to 1+0} \varphi(x) = \lim_{h \to 0} \varphi(1 + h) = \lim_{h \to 0} (1 + h) = 1;$$

$$\therefore \lim_{x \to 1-0} \varphi(x) = \lim_{x \to 1+0} \varphi(x).$$

Hence $\lim\limits_{x \to 1} \varphi(x)$ exists and it is equal to 1.

(iv) Meaning of $\lim\limits_{x \to \infty} f(x) = l$.

If, corresponding to any positive number ϵ, no matter how small, there exists a positive number N such that

$$| f(x) - l | < \epsilon,$$

provided only that $x > N$, where l is a fixed number, then l is called the limit of $f(x)$ as x tends to infinity.

This is expressed by writing

$$\lim_{x \to \infty} f(x) = l \quad \text{or} \quad f(x) \to l, \quad \text{as} \quad x \to \infty.$$

(v) Meaning of $\lim\limits_{x \to \infty} f(x) = \infty$.

If to any positive number M, no matter how great, there corresponds a positive number N such that

$$f(x) > M, \text{ provided only that } x > N,$$

we say that $f(x)$ tends to infinity with x, and we write

$$f(x) \to \infty \text{ as } x \to \infty.$$

Similarly the corresponding definitions of

'$f(x)$ tends to $-\infty$ as $x \to \infty$' can be given.

4.4. Fundamental Theorems

Suppose that as $x \to a$, $f(x) \to l$ and $g(x) \to l'$, then

(i) $f(x) + g(x) \to l + l'$,

(ii) $f(x) - g(x) \to l - l'$,

(iii) $f(x)g(x) \to ll'$,

(iv) $\dfrac{1}{f(x)} \to \dfrac{1}{l}$, *provided that* $l \neq 0$,

(v) $\dfrac{f(x)}{g(x)} \to \dfrac{l}{l'}$, *unless* $l' = 0$.

The proofs of the above theorems are left as exercises for the reader; the line of proof in each case is similar to those in the chapter "Real sequences".

4.5. Continuous Functions of a Real Variable

Let the curve C be the graph of a function $y = f(x)$, defined in the interval $a \leqslant x \leqslant b$.

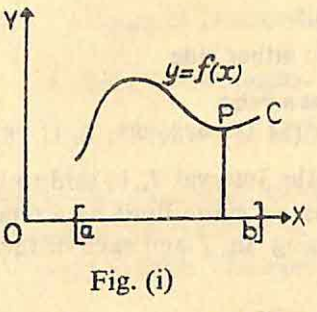


Fig. (i)

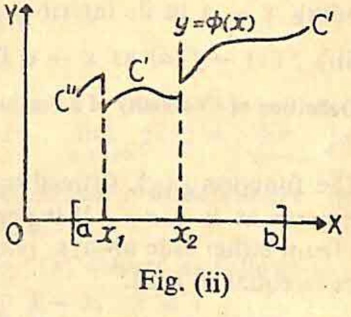


Fig. (ii)

In the graph we find that the curve C is not broken at any point. We say the graph is continuous in the interval $[a, b]$.

Let the curve C' [Fig. (ii)] be the graph of $y = \varphi(x)$, defined over the interval $a \leqslant x \leqslant b$. The curve C' is broken at $x = x_1$ and $x = x_2$ only. We say the graph is discontinuous at $x = x_1$, $x = x_2$.

The definition of continuity can be made in ordinary language as follows—

A function defined in the interval I is said to be continuous over I if its graph is a continuous curve; otherwise it is discontinuous.

Let us distinguish more precisely some of the properties which are involved in this definition.

In the first place it is evident that the property of the function $y = f(x)$ of which C is the graph may be analysed into some property possessed by the curve at each of its points. To be able to define continuity for all values of x we must first define continuity for any particular value of x. Let us therefore fix on some particular value of x, say the value $x = a$ corresponding to the point P of the graph.

The characteristic properties of $f(x)$ associated with this value of x are :

(i) $f(x)$ must be defined for $x = a$.

If $f(a)$ were not defined there would be a point missing from the curve.

(ii) $f(x)$ must be defined at all points of an interval including $x = a$ in its interior.

(iii) $f(x) \to f(a)$ as $x \to a$ from either side.

Definition of Continuity of a Function at a Point.

(M U 1978, '85; B. U. '85)

The function $f(x)$, defined over the interval I, is said to be continuous at $x = a \in I$ if it possesses a finite limit as x tends to a from either side always remaining in I and each of these limits is equal to $f(a)$.

Thus $f(x)$ is continuous at $x = a$ if

(i) $f(x)$ is defined at $x = a$ and

(ii) $\lim\limits_{x \to a+0} f(x) = \lim\limits_{x \to a-0} f(x) = f(a)$

i.e. $f(a + 0) = f(a - 0) = f(a)$

i.e. limit from right = limit from left = value of the function.

If a is an end point of the interval I of definition of $f(x)$ then one of the left-hand limit or right-hand limit exists. In that case, one that exists should be equal to $f(a)$.

Thus if a is a left-end point of the interval I then $f(x)$ is continuous at $x = a$ if $\lim\limits_{x \to a+0} f(x) = f(a)$ and if a is a right-end point of the interval I then $f(x)$ is continuous at $x = a$ if $\lim_{x \to a-0} f(x) = f(a)$.

This definition can be put in the distance form (or modulus form) as follows—

A function $f(x)$, defined over the interval I, is said to be continuous at $x = a \in I$, if given $\epsilon > 0$ there exists a positive number δ such that

$$|f(x) - f(a)| < \epsilon \text{ whenever } |x - a| < \delta \text{ and } x \in I$$

i.e. $f(a) - \epsilon < f(x) < f(a) + \epsilon$ whenever $x \in I$ such that

$$a - \delta < x < a + \delta.$$

A function $f(x)$ which is not continuous at $x = a$ is called *discontinuous* at $x = a$.

4.6. Types of Discontinuity

(i) Ordinary Discontinuity.

If $f(a+0) \neq f(a-0)$, i.e., $\lim_{x \to a+0} f(x) \neq \lim_{x \to a-0} f(x)$

then $f(x)$ is said to have *ordinary discontinuity* or discontinuity of first kind at $x = a$.

For example. The function $f(x)$ defined as follows—

$$f(x) = \begin{cases} 2 - x, & x \leqslant 1 \\ 3x, & x > 1 \end{cases}$$

has ordinary discontinuity at $x = 1$, for

$$\lim_{x \to 1-0} f(x) = 1 \neq 3 = \lim_{x \to 1+0} f(x).$$

(ii) Removable Discontinuity.

If $f(a+0) = f(a-0) \neq f(a)$ {or $f(a)$ is not defined},
i.e. $\lim_{x \to a+0} f(x) = \lim_{x \to a-0} f(x) \neq f(a)$ {or $f(a)$ is not defined}

then $f(x)$ is said to have *removable discontinuity* at $x = a$. It is called removable because by suitably defining $f(a)$, the function can be made continuous at $x = a$.

For example. The function $f(x) = \dfrac{x^2 - 4}{x - 2}$, $x \neq 2$ has a removable discontinuity at $x = 2$ for

$$\lim_{x\to 2+0} f(x) = 4 = \lim_{x\to 2-0} f(x) \neq f(2)$$

for at $x = 2$ the function is not defined.

Also the function $\varphi(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 3, & x = 2 \end{cases}$

has a removable discontinuity at $x = 2$ for

$$\lim_{x\to 2+0} \varphi(x) = \lim_{x\to 2-0} \varphi(x) = 4 \neq 3 = \varphi(2).$$

(iii) Infinite Discontinuity.

If either of $f(a+0)$, $f(a-0)$ or both are equal to $+\infty$ or $-\infty$, then $f(x)$ is said to have *infinite discontinuity* at $x = a$.

For example. The function $f(x) = \dfrac{2x}{(x-1)^2}$ has infinite discontinuity at $x = 1$ as $\lim\limits_{x\to 1+0} f(x) = \infty$.

(iv) Oscillatory Discontinuity.

If $f(x)$ is discontinuous at $x = a$ and the discontinuity is neither ordinary nor removable nor infinite, then the discontinuity is called *oscillatory discontinuity.* Such discontinuity may oscillate finitely or infinitely.

For example. The function $f(x) = \sin \dfrac{1}{x-a}$ has oscillatory discontinuity at $x = a$.

4.7. Continuity of a Function in an Interval

A function $f(x)$ defined over the interval I is said to be continuous in the interval $[a, b] \subseteq I$ if $f(x)$ is continuous at all points x such that $a \leqslant x \leqslant b$.

4.8. Differentiability at a Point (B. U. 1985; M. U. '85)

A function $f(x)$ defined in the interval I is said to be differentiable at $x = a \in I$ if $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists for all tendencies of x towards a so that x always remains in I.

In other words, the condition is that

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \text{ and } \lim_{h \to 0} \frac{f(a - h) - f(a)}{-h}$$

both exist and have the same definite value.

Thus the function $f(x)$ is said to be differentiable at $x = a$ if

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{f(a - h) - f(a)}{-h}$$

$= a$ definite number.

The value of the limit is called the derivative (differential co-efficient) of $f(x)$ at $x = a$ and it is denoted by

$$f'(a) \text{ or } Df(a).$$

If $x = a$ is an end-point of the domain of definition I then either of $\lim\limits_{h \to 0} \dfrac{f(a + h) - f(a)}{h}$, $\lim\limits_{h \to 0} \dfrac{f(a - h) - f(a)}{-h}$ can exist and that limit is called derivative of $f(x)$ at $x = a$.

If only $\lim\limits_{h \to 0} \dfrac{f(a + h) - f(a)}{h}$ exists then $f(x)$ is called differentiable from the right and the limit is called derivative from the right. It is denoted by $f'(a +)$ or $Rf'(a)$.

If only $\lim\limits_{h \to 0} \dfrac{f(a - h) - f(a)}{-h}$ exists then $f(x)$ is called differentiable from the left and the limit is called derivative from the left. It is denoted by $f'(a -)$ or $Lf'(a)$.

The definition of differentiability at a point in the distance form is as follows—

A function $f(x)$ defined in the interval I is said to be differentiable at $x = a \in I$, having the derivative l if given $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$\left| \frac{f(x) - f(a)}{x - a} - l \right| < \epsilon. \text{ whenever } |x - a| < \delta \text{ and } x \in I.$$

The number l is the derivative of $f(x)$ at $x = a$.

Theorem 4.1. *A function f is differentiable at $x = a$ if and only if there exists a number l such that*

$$f(a + h) - f(a) = lh + h.\eta$$

where η denotes a quantity which tends to 0 as $h \to 0$.

Proof. Let f be differentiable at $x = a$. Then there exists a number l such that $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} = l$.

Putting $x = a + h$,

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = l$$

or
$$\lim_{h \to 0} \left[\frac{f(a + h) - f(a)}{h} - l \right] = 0.$$

$$\therefore \quad \frac{f(a + h) - f(a)}{h} - l = \eta \text{ where } \eta \to 0 \text{ as } h \to 0.$$

$$\therefore \quad f(a + h) - f(a) = lh + h.\eta \text{ where } \eta \to 0 \text{ as } h \to 0.$$

Thus it is the necessary condition.

As the argument is reversible, the condition is also sufficient.

Theorem 4.2. *If a function is differentiable finitely at a point, then it must be continuous at that point.*

(Bh U 1967 H; R U '67 H; M U '66 H;
P U '85; B U '61 A, '66 H)

Proof. Let the function $f(x)$ be differentiable at $x = a$.

Then by definition $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$

$$= \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h} = A \text{ (say).}$$

From $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h} = A$, it follows, by the definition of limit, that, for a given number $\epsilon > 0$, we can find a number $\delta > 0$ such that

$$\left|\frac{f(a+h)-f(a)}{h} - A\right| \leqslant \epsilon, \text{ for } |h| < \delta$$

i.e. $| f(a+h)-f(a)-Ah | \leqslant \epsilon | h |$, for $| h | < \delta$.

But $| f(a+h)-f(a) | - | Ah | \leqslant | f(a+h)-f(a)-Ah |$.

$\therefore \quad | f(a+h)-f(a) | - | Ah | \leqslant \epsilon | h |$.

Hence $| f(a+h)-f(a) | \leqslant | h | (| A | + \epsilon)$.

Now, if $h \to +0$, then $f(a+h)-f(a) \to 0$

and also if $h \to -0$, then $f(a-h)-f(\text{a}) \to 0$.

Thus $f(a+0) = f(a) = f(a-0)$ and $f(a)$ is defined.

So $f(x)$ is continuous at $x = a$.

Note. *The converse of this theorem is not necessarily true, i.e. the condition of continuity is not sufficient for differentiability.*

i.e. the continuity of a function is a weaker condition than differentiability. (P U 1967 H)

Let us illustrate this by an example.

Consider the continuity and differentiability of the function $f(x) = | x |$ at $x = 0$.

(M U 1966 H, '78 A'85; Bh U '66 H; AMIE '81)

Test for continuity at $x = 0$.

Limit from right $= f(0+0) = \lim\limits_{h\to 0} f(0+h)$

$$= \lim_{h\to 0} | 0+h | = \lim_{h\to 0} | h | = 0$$

And limit from left $= f(0-0) = \lim\limits_{h\to 0} f(0-h)$

$$= \lim_{h\to 0} | 0-h | = \lim_{h\to 0} | h | = 0.$$

Since $f(0+0) = f(0-0) = f(0) = 0$,

so the given function is continuous at $x = 0$.

Test for differentiability at $x = 0$.

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|-0}{h} = 1, \text{ as } h > 0,$$

and $\displaystyle\lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{|0-h|-0}{-h}$

$$= \lim_{h\to 0} \frac{h}{-h} = -1.$$

Since these two limits are not equal, therefore the function is not differentiable at $x = 0$.

WORKED OUT EXAMPLES

1. *Prove that* $\displaystyle\lim_{x\to a} \frac{1}{1-e^{\frac{1}{x-a}}}$ *does not exist.*

Solution. Here $f(x) = \dfrac{1}{1-e^{\frac{1}{x-a}}}$.

Then $f(a+0) = \lim\limits_{h\to 0} f(a+h)$

$$= \lim_{h\to 0} \frac{1}{1-e^{\frac{1}{a+h-a}}}$$

$$= \lim_{h\to 0} \frac{1}{1-e^{\frac{1}{h}}}$$

$$= \frac{1}{1-e^{\infty}} = \frac{1}{1-\infty} = 0$$

and $f(a-0) = \lim\limits_{h\to 0} f(a-h)$

$$= \lim_{h\to 0} \frac{1}{1-e^{\frac{1}{a-h-a}}}$$

$$= \lim_{h\to 0} \frac{1}{1 - e^{-\frac{1}{h}}}$$

$$= \frac{1}{1-0} = 1.$$

Since $f(a+0) \neq f(a-0)$, therefore the given limit does not exist.

2. *Given that* $f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0 \\ 2, & x = 0. \end{cases}$

Test the continuity of $f(x)$ *at* $x = 0$. (P U 1980; B U '80)

Solution. $f(x)$ is continuous at $x = 0$ if

$$\lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(0-h) = f(0).$$

Here $\lim\limits_{h\to 0} f(0+h) = \lim\limits_{h\to 0} \dfrac{\sin(0+h)}{0+h} = \lim\limits_{h\to 0} \dfrac{\sin h}{h} = 1,$

$$\lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{\sin(0-h)}{0-h} = \lim_{h\to 0} \frac{\sin h}{h} = 1;$$

$$f(0) = 2.$$

$\therefore \lim\limits_{h\to 0} f(0+h) = \lim\limits_{h\to 0} f(0-h) \neq f(0).$

$\therefore$ $f(x)$ is discontinuous at $x = 0$.

Note. The type of discontinuity of $f(x)$ at $x = 0$ in the above problem is a removable discontinuity.

If $f(0)$ is defined as 1 instead of 2 then the function becomes continuous at $x = 0$.

3. *If* $f(x) = \dfrac{x}{1 + e^{\frac{1}{x}}}$, $x \neq 0$, and $f(0) = 0$, *show that* $f(x)$ *is not differentiable* at $x = 0$.

Solution. Here $\underset{h\to 0}{\text{Lt}}\ \dfrac{f(0+h)-f(0)}{h}$

$$= \underset{h\to 0}{\text{Lt}}\ \frac{\dfrac{0+h}{1+e^{\frac{1}{0+h}}}-0}{h} = \underset{h\to 0}{\text{Lt}}\ \frac{\dfrac{h}{1+e^{\frac{1}{h}}}}{h}$$

$$= \underset{h\to 0}{\text{Lt}}\ \frac{1}{1+e^{\frac{1}{h}}} = \frac{1}{1+e^{\infty}} = \frac{1}{\infty} = 0,$$

and $\underset{h\to 0}{\text{Lt}}\ \dfrac{f(0-h)-f(0)}{-h} = \underset{h\to 0}{\text{Lt}}\ \dfrac{\dfrac{0-h}{1+e^{\frac{1}{0-h}}}-0}{-h}$

$$= \underset{h\to 0}{\text{Lt}}\ \frac{\dfrac{-h}{1+e^{-\frac{1}{h}}}}{-h} = \underset{h\to 0}{\text{Lt}}\ \frac{1}{1+e^{-\frac{1}{h}}} = \underset{h\to 0}{\text{Lt}}\ \frac{1}{1+\dfrac{1}{e^{\frac{1}{h}}}}$$

$$= \frac{1}{1+\dfrac{1}{e^{\infty}}} = \frac{1}{1+0} = 1.$$

Since the above two limits are different, hence the function is not differentiable at $x = 0$.

4. *Is the following function continuous and differentiable at* $x = \frac{1}{2}$?

$$f(x) = \begin{cases} x, & 0 \leqslant x < \frac{1}{2} \\ 1-x, & \frac{1}{2} \leqslant x < 1. \end{cases}$$ (P U 1973)

Solution. $f(x)$ is continuous at $x = \frac{1}{2}$ if

$$\lim_{h\to 0} f(\tfrac{1}{2}+h) = \lim_{h\to 0} f(\tfrac{1}{2}-h) = f(\tfrac{1}{2}).$$

Here $\lim\limits_{h\to 0} f(\frac{1}{2}+h) = \lim\limits_{h\to 0} \{1-(\frac{1}{2}+h)\} = \frac{1}{2}$,

$\lim\limits_{h\to 0} f(\frac{1}{2}-h) = \lim\limits_{h\to 0} \{\frac{1}{2}-h\} = \frac{1}{2}$,

$f(\frac{1}{2}) = 1 - \frac{1}{2} = \frac{1}{2}$;

$\therefore \quad \lim\limits_{h\to 0} f(\frac{1}{2}+h) = \lim\limits_{h\to 0} f(\frac{1}{2}-h) = f(\frac{1}{2}).$

$\therefore$ $f(x)$ is continuous at $x = \frac{1}{2}$.

$f(x)$ is differentiable at $x = \frac{1}{2}$ if

$$\lim_{h\to 0} \frac{f(\frac{1}{2}+h) - f(\frac{1}{2})}{h} = \lim_{h\to 0} \frac{f(\frac{1}{2}-h) - f(\frac{1}{2})}{-h}.$$

$$\text{Here } \lim_{h\to 0} \frac{f(\frac{1}{2}+h) - f(\frac{1}{2})}{h} = \lim_{h\to 0} \frac{1 - (\frac{1}{2}+h) - \frac{1}{2}}{h}$$

$$= \lim_{h\to 0} \frac{-h}{h} = \lim_{h\to 0} -1 = -1.$$

$$\lim_{h\to 0} \frac{f(\frac{1}{2}-h) - f(\frac{1}{2})}{-h} = \lim_{h\to 0} \frac{\frac{1}{2} - h - \frac{1}{2}}{-h}$$

$$= \lim_{h\to 0} \frac{-h}{-h} = \lim_{h\to 0} 1 = 1.$$

$\therefore$ The two limits are not equal.

$\therefore$ $f(x)$ is not differentiable at $x = \frac{1}{2}$.

5. *If* $f(x) = x^2 \sin \frac{1}{x}$, *for* $x \neq 0$, *and* $f(0) = 0$,

show that $f(x)$ *is continuous and differentiable at* $x = 0$ *but its derivative is not continuous there.*

(R U 1962 H, '64 H; B U '65 H; M U '64 H, '80 H, '84; P. U. '79)

Solution. Test for continuity at $x = 0$.

$$f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} (0+h)^2 \sin \frac{1}{0+h}$$

$$= \lim_{h\to 0} h^2 \sin \frac{1}{h} = 0.$$

$$\text{Again, } f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} (0-h)^2 \sin \frac{1}{0-h}$$

$$= \lim_{h\to 0} h^2 \sin\left(-\frac{1}{h}\right) = 0.$$

Since $f(0+0) = f(0-0) = f(0)$, therefore the function is continuous at $x = 0$.

To test for differentiability at $x = 0$.

$$\text{We have } \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{(0+h)^2 \sin \frac{1}{0+h} - 0}{h}$$

$$= \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0,$$

$$\text{and } \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{(0-h)^2 \sin \frac{1}{0-h} - 0}{-h}$$

$$= \lim_{h \to 0} \frac{-h^2 \sin \frac{1}{h}}{h} = \lim_{h \to 0} \left(-h \sin \frac{1}{h}\right) = 0.$$

Since the above two limits are equal therefore the function is differentiable at $x = 0$ and $f'(0) = 0$. .. (1)

$$\text{Again } f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 \sin \frac{1}{x+h} - x^2 \sin \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \left[\frac{x^2}{h} \left\{ \sin \frac{1}{x+h} - \sin \frac{1}{x} \right\} + \left\{ 2x \sin \frac{1}{x+h} + h \sin \frac{1}{x+h} \right\} \right]$$

$$= \lim_{h \to 0} \frac{x^2}{h} . 2 \cos \frac{\left(\frac{1}{x+h} + \frac{1}{x}\right)}{2} . \sin \frac{\frac{1}{x+h} - \frac{1}{x}}{2} + 2x \sin \frac{1}{x} + 0. \sin \frac{1}{x}$$

$$= x^2, 2 \cos \frac{1}{x} . \lim_{h \to 0} \sin \frac{-h}{2x(x+h)} \Big/ h + 2x \sin \frac{1}{x}$$

$$= 2x^2 \cos \frac{1}{x} . \lim_{h\to 0} \frac{- \sin \dfrac{h}{2x(x+h)}}{\dfrac{h}{2x(x+h)}}$$

$$\times \frac{1}{2x(x+h)} + 2x \sin \frac{1}{x}$$

$$= 2x^2 \cos \frac{1}{x} . \left(- \frac{1}{2x^2} \right) + 2x \sin \frac{1}{x}$$

$$= - \cos \frac{1}{x} + 2x \sin \frac{1}{x} \text{ for all } x \neq 0.$$

Now $f'(x) = \varphi(x)$, (say) is continuous at $x = 0$ if

$$\lim_{h\to 0} \varphi (0 + h) = \lim_{h\to 0} \varphi (0 - h) = \varphi (0).$$

Here $\lim_{h\to 0} \varphi (0 + h) = \lim_{h\to 0} f'(0 + h)$

$$= \lim_{h\to 0} \left[- \cos \frac{1}{0+h} + 2(0 + h). \sin \frac{1}{0+h} \right] = -1,$$

$$\lim_{h\to 0} \varphi (0 - h) = \lim_{h\to 0} f'(0 - h)$$

$$= \lim_{h\to 0} \left[- \cos \frac{1}{0-h} + 2(0 - h) \sin \frac{1}{0-h} \right] = -1,$$

$\varphi (0) = f'(0) = 0$, by (1).

$\therefore$ $f'(x)$ is not continuous at $x = 0$.

6. *Discuss the continuity of* $\sqrt{\{(x - a)(b - x)\}}$.

(B U 1975)

Solution. Let $a < b$.

For $a < x < b$, $x - a > 0$ and $b - x > 0$.

$\therefore$ $f(x) = \sqrt{\{(x - a)(b - x)\}}$ is defined.

Also $f(a) = 0$

$f(b) = 0.$

If $x > b, x - a > 0$ and $b - x < 0$.

$\therefore$ $f(x)$ is not defined for $x > b$.

If $x < a, x - a < 0$ and $b - x > 0$.

$\therefore$ $f(x)$ is not defined for $x < a$.

Thus $f(x)$ is defined in the closed interval $[a, b]$ as follows—

$$f(x) = \begin{cases} 0, & x = a \\ \sqrt{\{(x-a)(b-x)\}}, & a < x < b \\ 0, & x = b. \end{cases}$$

Now $\lim\limits_{h \to 0} f(a+h) = \lim\limits_{h \to 0} \sqrt{\{(a+h-a)(b-a-h)\}} = 0,$

$f(a) = 0;$

$\therefore$ $\lim\limits_{h \to 0} f(a+h) = f(a)$. So $f(x)$ is continuous at $x = a$ as it is the left end point of the domain of definition.

Again $\lim\limits_{h \to 0} f(b-h) = \lim\limits_{h \to 0} \sqrt{\{(b-h-a)(b-b+h)\}}$

$$= 0 = f(b).$$

$\therefore$ $f(x)$ is continuous at $x = b$ as it is the right end point of the domain of definition.

Let $a < y < b$. Then $y = a + (b-a)\theta$ where $0 < \theta < 1$.

Now
$$\begin{aligned}\lim_{h \to 0} f(y \pm h) &= \lim_{h \to 0} \sqrt{\{(y \pm h - a)(b - y \mp h)\}} \\ &= \lim_{h \to 0} \sqrt{[\{a + (b-a)\theta \pm h - a\}} \\ &\qquad \times \{b \mp h - a - (b-a)\theta\}] \\ &= \lim_{h \to 0} \sqrt{[\{(b-a)\theta \pm h\}} \\ &\qquad \times \{(b-a)(1-\theta) \mp h\}] \\ &= \sqrt{[(b-a)\theta . (b-a)(1-\theta)]} \\ &= (b-a)\sqrt{\theta(1-\theta)}.\end{aligned}$$

$$\begin{aligned}f(y) &= f\{a + (b-a)\theta\} \\ &= \sqrt{[\{a + (b-a)\theta - a\}\{b - a - (b-a)\theta\}]} \\ &= \sqrt{[(b-a)\theta . \{(b-a)(1-\theta)\}]} \\ &= (b-a)\sqrt{\theta(1-\theta)}.\end{aligned}$$

$\therefore$ $f(x)$ is continuous at all points such that $a < x < b$.

$\therefore$ $f(x)$ is continuous in the closed interval $[a, b]$.

7. *A function y is defined as follows :*

"y is the smallest positive number that makes $x + y$ *an integer". For what values of x is the function continuous ?*

Solution. The given statement can be shown by the following functional relation :

$$x + y = \alpha \quad \text{when } \alpha - 1 < x < \alpha$$

i.e. if α is the nearest integer to x.

$$x + y = \alpha + 1 \quad \text{when } \alpha < x < \alpha + 1$$

i.e. if $\alpha + 1$ is the nearest integer to x.

Thus $y = f(x)$ is defined as follows—

$$\begin{aligned} f(x) &= \alpha - x \quad , \quad \alpha - 1 < x < \alpha \\ &\quad \alpha + 1 - x \; , \quad \alpha < x < \alpha + 1 \end{aligned}$$

where α is an integer.

Now, $\lim\limits_{h\to0} f(\alpha - h) = \lim\limits_{h\to0} \{\alpha - (\alpha - h)\} = 0,$

$\lim\limits_{h\to0} f(\alpha + h) = \lim\limits_{h\to0} \{\alpha + 1 - (\alpha - h)\} = 1;$

$\therefore \quad \lim\limits_{h\to0} f(\alpha - h) \neq \lim\limits_{h\to0} f(\alpha + h)$ for every integer α.

$\therefore \quad f(x)$ is discontinuous for $x = \alpha$ where α is any integer.

Also it is clear by definition that $f(x)$ is continuous for values of $\alpha - 1 < x < \alpha$ where α is any integer.

$\therefore$ The function y is continuous for all values of x that are not integers.

Problem-Set 7

1. Can the value of $\lim\limits_{h\to0} \dfrac{|x|}{x}$ be obtained ? Give reasons.

2. A function $f(x)$ is defined as follows—

$$f(x) = \begin{cases} x \; , & x < 0 \\ 1 \; , & x = 0 \\ x^2 \; , & x > 0. \end{cases}$$

Does $\lim\limits_{x\to0} f(x)$ exist ? Verify.

3. (i) If $f(x) = x \sin\dfrac{1}{x}$, $x \neq 0$ and $f(0) = 1$, show that $f(x)$ is discontinuous at $x = 0$. (R U 1976; B P S C '83)

(ii) Is the function $f(x)$ defined by $f(x) = \dfrac{\sin 2x}{x}$ when $x \neq 0$ and $f(0) = 4$ continuous at $x = 0$? (P. U. 1972)

4. (i) Discuss the continuity of $f(x)$ at $x = 0, 1, 2$ where

$$f(x) = \begin{cases} -x^2, & x \leqslant 0 \\ 5x - 4, & 0 < x \leqslant 1 \\ 4x^2 - 3x, & 1 < x < 2 \\ 3x + 4, & x \geqslant 2. \end{cases}$$

(ii) A function $f(x)$ is defined as follows : $f(0)=0$, $f(x) = \frac{1}{2} - x$, when $0 < x < \frac{1}{2}$ and $f(\frac{1}{2}) = \frac{1}{2}$. Discuss the continuity at $x=0$ and at $x=\frac{1}{2}$. (P. U. 1966, '75)

5. Prove that the function

$$f(x) = \frac{x^2}{a} - a, \quad 0 < x < a$$

$$f(x) = 0 \quad , \quad x = a$$

$$f(x) = a - \frac{a^3}{x^2}, \quad x > a$$

is continuous at $x = a$. (P. U. 1985)

6. Prove that $2^{\frac{1}{x}}$ is discontinuous at $x = 0$.

7. (i) Examine the continuity at $x=0$ and $x=1$ of the following function—

$$\begin{aligned} f(x) &= 1, & x &= 0 \\ &= x, & 0 &< x < 1 \\ &= 0, & x &= 1. \end{aligned}$$

(P U 1976)

(ii) Show that the function $f(x)$ where

$$f(x) = \begin{cases} x, \text{ when } 0 < x < \frac{1}{2} \\ 1, \text{ when } x = \frac{1}{2} \\ 1 - x, \text{ when } \frac{1}{2} < x < 1 \end{cases}$$

is discontinuous at $x = \frac{1}{2}$. (P. U. 1967, '70)

8. A function $\varphi(x)$ is defined as follows—

$$\varphi(x) = \frac{x^2 - a^2}{x - a} \text{ when } 0 \leqslant x < a$$

$$\varphi(x) = 2a \text{ when } x \geqslant a.$$

Discuss if $\varphi(x)$ is continuous at $x = a$. (M U 1975; P U '66)

9. Prove that the function $f(x) = \dfrac{e^{1/x}}{1 + e^{1/x}}, x \neq 0$ and $f(0) = 0$ is discontinuous at $x = 0$.

10. (i) Is the function $f(x) = \dfrac{3x + 4 \tan x}{x}$ continuous at $x = 0$? If not, how may the function be defined to make it continuous at this point. Give reasons for your answer.

(ii) Prove that the function $f(x)$ where $f(x)=\frac{x^2-4}{x-2}$ when $x \neq 2$, $f(2)=6$ is discontinuous at $x=2$. What value may be assigned to $f(x)$ at $x=2$ so that $f(x)$ becomes continuous at $x=2$? (P. U. 1971)

11. Is the following function continuous and differentiable at $x = 0$?

(i) $$f(x) = \begin{cases} 3+2x, & -\frac{3}{2} < x \leqslant 0 \\ 3-2x, & 0 < x < \frac{3}{2}. \end{cases}$$

(ii) $f(x) = x \sin \frac{1}{x}, x \neq 0$

$f(0) = 0.$ (M U 1977, '79; P U '77; A M I E '74, '80; Mith. U. '85; B. U. '85)

12. If $$f(x) = \begin{cases} x^2 \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0, \end{cases}$$

find $f'(0)$ if possible.

13. If $f(x) = x - 1$ when $x > 1$
$= 0$ when $x = 1$
$= 1 - x$ when $x < 1$ i.e. $f(x) = |x - 1|$,

show that $f(x)$ is continuous at $x = 1$, but not differentiable there. (A M I E 1976)

14. (i) Let $f(x) = e^{-\frac{1}{x^2}} \sin \frac{1}{x}, x \neq 0$

$f(0) = 0.$

Test the differentiability of the function at $x = 0$.

(ii) A function is defined as follows—

$$\varphi(x) = 1 + \sin x, \quad 0 \leqslant x < \frac{\pi}{2}$$
$$\varphi(x) = 2 + \left(x - \frac{\pi}{2}\right)^2, \quad \frac{\pi}{2} \leqslant x < \infty.$$

Discuss its differentiability at $x = \frac{\pi}{2}$.

15. Discuss the continuity of the following function for real values of x.

(i) $x^3 - 3x$, (ii) $\sqrt{5x - x^2 - 6}$.

16. The function $f(x)$ is defined as follows—

$$\begin{aligned} f(x) &= \tfrac{3}{2}a^2, \quad 0 \leqslant x \leqslant a \\ &= 2a^2 - \frac{1}{6}x^2 - \frac{1}{3}\cdot\frac{a^3}{x}, \; a < x \leqslant 2a \\ &= \frac{7}{3}\cdot\frac{a^3}{x}, \; x > b. \end{aligned}$$

Show that $f'(x)$ is continuous for every positive value of x.

17. Let $f(x) = \begin{cases} (x-1)^2 \sin \dfrac{1}{1-x} - |x| & \text{when } x \neq 1 \\ -1 \text{ when } & x = 1, \end{cases}$

be a real valued function. Find the set of points where $f(x)$ is not differentiable.

18. If $f(x) = 1 - x^2$ for $x \leqslant 0$
$= \cos x$ for $x > 0$,

show that

(i) $f(x)$ is continuous in $-1 \leqslant x \leqslant \dfrac{\pi}{2}$;

(ii) $f(x)$ is differentiable in $-1 < x < \dfrac{\pi}{2}$. (A M I E 1980).

19. Show that $\varphi(x) = |x| + |x-1|$ is continuous but not derivable for $x = 0, x = 1$.

4.9. Properties of Continuous Function

Theorem 4.3. *If $f(x)$ and $\varphi(x)$ are continuous at $x = a$ then*

(i) *$f(x) \pm \varphi(x)$ are continuous at $x = a$;*

(ii) *$f(x) \times \varphi(x)$ is continuous at $x = a$;*

(iii) *$\dfrac{f(x)}{\varphi(x)}$ is continuous at $x = a$ provided*

$$\varphi(x) \neq 0 \text{ for } a - h \leqslant x \leqslant a + h.$$

Proof. As $f(x)$ and $\varphi(x)$ are continuous at $x = a$,

$$\lim_{x \to a} f(x) = f(a) \text{ and } \lim_{x \to a} \varphi(x) = \varphi(a).$$

(i) Let $F(x) = f(x) \pm \varphi(x)$; then $F(a) = f(a) \pm \varphi(a)$.

Now
$$\begin{aligned} \lim_{x \to a} F(x) &= \lim_{x \to a} [f(x) \pm \varphi(x)] \\ &= \lim_{x \to a} f(x) \pm \lim_{x \to a} \varphi(x), \quad \text{\{by theorem on limit\}} \\ &= f(a) \pm \varphi(a) = F(a). \end{aligned}$$

$\therefore$ $F(x)$ is continuous at $x = a$.

(ii) Let $G(x) = f(x) \times \varphi(x)$; then $G(a) = f(a) \times \varphi(a)$.

Now $\lim\limits_{x\to a} G(x) = \lim\limits_{x\to a} [f(x) \times \varphi(x)]$

$= \lim\limits_{x\to a} f(x) \times \lim\limits_{x\to a} \varphi(x)$. {by theorem on limit}

$= f(a) \times \varphi(a) = G(a)$.

$\therefore$ $G(x)$ is continuous at $x = a$.

(iii) Let $G(x) = \dfrac{f(x)}{\varphi(x)}$; then $G(a) = \dfrac{f(a)}{\varphi(a)}$.

Not $\lim\limits_{x\to a} G(x) = \lim\limits_{x\to a} \dfrac{f(x)}{\varphi(x)}$

$= \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} \varphi(x)}$, {by theorem on limit}

$= \dfrac{f(a)}{\varphi(a)} = G(a)$.

$\therefore$ $G(x)$ is continuous at $x = a$.

Note. (a) If one of $f(x)$, $\varphi(x)$ is continuous and the other discontinuous at $x = a$ then $f(x) \pm \varphi(x)$ will be discontinuous at $x = a$.

(b) The properties (i), (ii), (iii) can be extended for a finite number of functions.

(c) By the above theorem it is obvious that a *polynomial function* i.e. a function of the type $a_0x^n+a_1x^{n-1}+\dots+a_n$, $n\epsilon N$, $a_i\epsilon R$, is continuous at all points ϵR.

Theorem 4.4. *If $f(x)$ be continuous in the closed interval $[a, b]$, then, given ϵ, the interval can always be divided up into a finite number of sub-intervals such that $|f(x_1) - f(x_2)| < \epsilon$, where x_1 and x_2 are any two points in the same sub-interval.*

Proof. Suppose that the theorem is not true.

Let c be the mid-point of $[a, b]$. Then $[a, b]$ is divided into two equal sub-intervals $[a, c]$ and $[c, b]$.

The theorem must not be true in at least one of the two sub-intervals $[a, c]$ and $[c, b]$.

Suppose it is not true in $[c, b]$. Denote this sub-interval by $[a_1, b_1]$. It is evident that the interval $[a_1, b_1]$ lies wholly inside $[a, b]$ and is of length $b_1 - a_1$, that is, $\frac{1}{2}(b - a)$.

Again divide $[a_1, b_1]$ into two equal sub-intervals. We denote the interval in which the theorem is not true by $[a_2, b_2]$. Obviously the sub-interval $[a_2, b_2]$ lies wholly inside $[a_1, b_1]$ and is of length $b_2 - a_2$, that is, $\frac{1}{2}(b_1 - a_1)$, that is,

$$\tfrac{1}{2}.\tfrac{1}{2}(b-a) \Rightarrow \frac{1}{2^2}(b-a).$$

Apply this process of repeated bisection. In this way we get an interval $[a_n, b_n]$ in which the theorem is not true and this interval lies wholly inside the preceding interval $[a_{n-1}, b_{n-1}]$ and is of length $b_n - a_n$, that is, $\frac{1}{2^n}(b-a)$.

$$\therefore \quad \lim_{n\to\infty} (b_n - a_n) = \lim_{n\to\infty} \frac{b-a}{2^n} = 0$$

$$\Rightarrow \quad \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = x_0 \text{ (say).}$$

Suppose, for definiteness, that x_0 does not coincide with a or b.

Since $f(x)$ is continuous at $x = x_0$, therefore, by definition of continuity, there exists a value of δ such that

$$|f(x) - f(x_0)| < \frac{\epsilon}{2}, \text{ when } |x - x_0| < \delta. \qquad .. \quad (1)$$

If n be chosen so large that $b_n - a_n$ is less than δ, then the interval $[a_n, b_n]$ is contained entirely within the interval

$$[x_0 - \delta, x_0 + \delta].$$

Let x_1 and x_2 be any two points in (a_n, b_n), then from (1), we get

$$|f(x_1) - f(x_0)| < \frac{\epsilon}{2}$$

$$\text{and} \quad |f(x_2) - f(x_0)| < \frac{\epsilon}{2}.$$

Now $f(x_1) - f(x_2) = f(x_1) - f(x_0) + f(x_0) - f(x_2)$

$$\Rightarrow \quad |f(x_1) - f(x_2)| = |\{f(x_1) - f(x_0)\} + \{f(x_0) - f(x_2)\}|$$

$$\Rightarrow \quad |f(x_1) - f(x_2)| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$\Rightarrow |f(x_1) - f(x_2)| < \epsilon.$$

This is a contradiction to our supposition. Hence our supposition is wrong. In other words, the theorem must be true.

Theorem 4.5. *Prove that a function, which is continuous in a closed interval* $[a, b]$, *is bounded therein.* (Mith. U. 1985)

Proof. We know that if $f(x)$ be continuous in the closed interval $[a, b]$, then, given ϵ, the interval can always be divided up into a finite number of sub-intervals such that

$$|f(x_1) - f(x_2)| < \epsilon, \quad \ldots (1)$$

where x_1 and x_2 are any two points in the same sub-interval.

Let the dividing points be $x_0 = a, x_1, x_2, \ldots, x_{n-1}, x_n = b$.

Let x be any point in the first sub-interval $[a, x_1]$.

Then, from (1), we have

$$|f(a) - f(x)| < \epsilon. \quad \ldots (2)$$

$$\begin{aligned}
\text{Now}\quad & f(x) = f(a) + \{f(x) - f(a)\} \\
\Rightarrow\quad & |f(x)| = |f(a) + \{f(x) - f(a)\}| \\
\Rightarrow\quad & |f(x)| \leqslant |f(a)| + |f(x) - f(a)| \\
\Rightarrow\quad & |f(x)| < |f(a)| + \epsilon, \text{ using (2).}
\end{aligned}$$

In particular, when $x = x_1$,

$$|f(x_1)| < |f(a)| + \epsilon. \quad \ldots (3)$$

Again, let x be any point in the second sub-interval $[x_1, x_2]$.

Then from (1), we have

$$|f(x_1) - f(x)| < \epsilon. \quad \ldots (4)$$

$$\begin{aligned}
\text{Now}\quad & f(x) = f(x_1) + \{f(x) - f(x_1)\} \\
\Rightarrow\quad & |f(x)| = |f(x_1) + \{f(x) - f(x_1)\}| \\
\Rightarrow\quad & |f(x)| \leqslant |f(x_1)| + |f(x) - f(x_1)| \\
\Rightarrow\quad & |f(x)| < |f(x_1)| + \epsilon, \text{ from (4)} \\
\Rightarrow\quad & |f(x)| < |f(a)| + 2\epsilon, \text{ using (3).}
\end{aligned}$$

In particular, when $x = x_2$,

$$|f(x_2)| < |f(a)| + 2\epsilon.$$

By proceeding in this way we get, when x is any point in n^{th} sub-interval $[x_{n-1}, b]$,

$$|f(x)| < |f(a)| + n\epsilon.$$

This inequality is true for the whole interval $[a, b]$, that is, all the values of $f(x)$ in the interval $[a, b]$ lie between $f(a) - n\epsilon$ and $f(a) + n\epsilon$.

Hence $f(x)$ is bounded in $[a, b]$.

Theorem 4.6. *If the function $f(x)$ is continuous in the closed interval $[a, b]$, then prove that $f(x)$ attains its bounds at least once in $[a, b]$.* (M. U. 1985)

Proof. By the previous theorem $f(x)$ is bounded in $[a, b]$. Let the least upper and greatest lower bounds of $f(x)$ in the closed interval $[a, b]$ be M and m respectively.

Then we have to prove that there are at least two points x_1 and x_2 in $[a, b]$ such that $f(x_1) = M$ and $f(x_2) = m$.

Let $f(x_1) \neq M$, that means $f(x)$ does not attain M at any point of $[a, b]$.

$\therefore\ M - f(x) \neq 0$ at any point of $[a, b]$.

Since $f(x)$ is continuous in $[a, b]$, therefore $M - f(x)$ is also continuous in $[a, b]$.

$\therefore\ \dfrac{1}{M - f(x)}$ must be continuous in $[a, b]$, as $M - f(x) \neq 0$.

We know that a function which is continuous in an interval is bounded therein.

$\therefore\ \dfrac{1}{M - f(x)}$ must be bounded in $[a, b]$.

If $G > 0$ be its least upper bound, then

$$\frac{1}{M - f(x)} \leqslant G$$

$$\Rightarrow\quad M - f(x) \geqslant G$$

$$\Rightarrow\quad f(x) \leqslant M - \frac{1}{G}.$$

This relation is not true, as M is the least upper bound of $f(x)$ in $[a, b]$.

Hence our supposition is wrong. Thus $f(x)$ attains its least upper bound M at least once in $[a, b]$.

Similarly we can prove that $f(x)$ must attain its greatest lower bound m at least once in $[a, b]$.

This completes the proof of the theorem.

Theorem 4.7. *A continuous function which has opposite signs at two points vanishes at least once between these points, that is, if $f(x)$ be continuous in the closed interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then there is at least one value of x between a and b for which $f(x) = 0$.*

Proof. For the sake of definiteness, let us suppose that

$$f(a) < 0 \text{ and } f(b) > 0.$$

Since $f(x)$ is continuous and $f(a) < 0$, therefore $f(x)$ will be negative in the neighbourhood of a.

Again since $f(x)$ is continuous and $f(b) > 0$, therefore $f(x)$ will be positive in the neighbourhood of b.

The set of values of x between a and b which make $f(x)$ positive is bounded below by a, and hence possesses an exact lower bound k.

Hence $a < k < b$.

In this way we find that $f(x)$ is positive in the interval $k < x < b$ and is negative or zero in the interval $a \leqslant x < k$.

Since $f(x)$ is continuous at $x = k$, therefore, by the definition of continuity, $f(k - 0) = f(k) = f(k + 0)$.

Since $f(x)$ is negative or zero in $a \leqslant x < k$, therefore $f(k - 0)$ must be negative or zero and, therefore, $f(k)$, which is equal to $f(k - 0)$, must be negative or zero.

We shall now show that $f(k)$ cannot be negative.

Since $f(x)$ is positive in the interval $k < x < b$, that is, $b > x > k$, therefore $f(k + 0)$ cannot be negative and since $f(k) = f(k + 0)$, therefore $f(k)$ cannot be negative.

Hence it follows that $f(k) = 0$ and the theorem is therefore proved.

Note. (i) This property of continuous function is of great importance in locating real roots of an equation.

Example. Consider the equation $2x^3 - x^2 - 2x + 3 = 0$.

Let $f(x) = 2x^3 - x^2 - 2x + 3$.

Clearly $f(x)$ is continuous for real values of x.

Again $f(-1) = -2 - 1 + 2 + 3 = 2 > 0$

$$f(-2) = -16 - 4 + 4 + 3 = -13 < 0.$$

$\therefore$ $f(-2)$ and $f(-1)$ are of opposite signs.

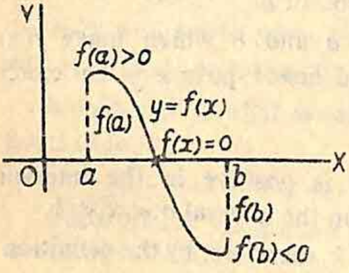
$\therefore$ $f(x) = 0$ is satisfied for at least once between

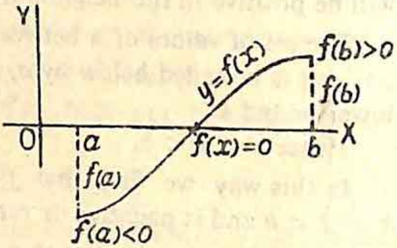
$$-2 \text{ and } -1.$$

Thus we can say, the equation has at least one real root lying between -2 and -1.

(ii) Geometrical meaning of the above property of a continuous function can be understood as follows—

If the graph of the function $y = f(x)$ is continuous from $x = a$ to $x = b$ and $f(a)$ and $f(b)$ are of opposite signs then the graph must intersect the x-axis at some point between $x = a$ and $x = b$ at least once.





Theorem 4.8. *If $f(a)$ and $f(b)$ are unequal, and $f(x)$ is cotinuous in the closed interval $[a, b]$, then $f(x)$ assumes at least once every value between $f(a)$ and $f(b)$.*

Proof. Let G be any number between $f(a)$ and $f(b)$. that is, $f(a) < G < f(b) \Rightarrow f(a) - G < 0$ and $f(b) - G > 0$.

Then we have to prove that $f(x) = G$.

Since $f(x)$ is continuous in $[a, b]$, therefore $f(x) - G$ must be continuous in $[a, b]$.

Also since the values of $f(x) - G$ at $x = a$ and b, that is, $f(a) - G$ and $f(b) - G$ have opposite signs, therefore $f(x) - G$ must vanish at least once between a and b.

Hence $f(x) - G = 0 \Rightarrow f(x) = G$.

But G is any value between $f(a)$ and $f(b)$.

Therefore $f(x)$ takes all values at least once between $f(a)$ and $f(b)$.

Problem Set 8

1. Using properties of continuous function show that the equation $x^5 - 3x = 1$ has at least one root lying between 1 and 2.

2. Show that the equation $\frac{a_1}{x-\lambda_1} + \frac{a_2}{x-\lambda_2} + \frac{a_3}{x-\lambda_3} = 0$, where $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $\lambda_1 < \lambda_2 < \lambda_3$ has two real roots lying in the internal (λ_1, λ_2) and (λ_2, λ_3).

ANSWERS

Problem Set 1

2. (i) 3; (ii) 3.

Problem Set 2

1. integer greater than $\sqrt{\frac{3}{4\epsilon}-\frac{5}{2}}$. 4. 0.

Problem Set 3

1. (i) cgt.; (ii) cgt.; (iii) div.; (iv) div.
2. (i) div.; (ii) cgt.; (iii) div.; (iv) cgt.;
   (v) div.; (vi) cgt.
3. (i) cgt.; (ii) div.; (iii) cgt.; (iv) div.
4. (i) div.; (ii) cgt.; (iii) cgt.; (iv) cgt. (v) div.
5. (i) cgt.; (ii) div.
6. (i) cgt. if $p > 1$, div. if $p \leqslant 1$;
   (ii) cgt. if $p > \frac{1}{2}$, div. if $p \leqslant \frac{1}{2}$.
7. (i) cgt.; (ii) cgt.; (iii) div.

Problem Set 4

1. cgt. 2. (i) cgt.; (ii) cgt.
3. (i) cgt. if $x \leqslant 1$, div. if $x > 1$;
   (ii) cgt. if $x < 1$, div. if $x \geqslant 1$;
   (iii) cgt. if $x < 1$, div. if $x \geqslant 1$;
   (iv) cgt. if $x < 1$, div. if $x \geqslant 1$;
   (v) cgt. if $x \leqslant 1$, div. if $x > 1$;
   (vi) cgt. if $x < 1$, div. if $x \geqslant 1$;
   (vii) cgt. if $x \leqslant 1$, div. if $x > 1$;
   (viii) cgt. if $x < 1$, div. if $x \geqslant 1$;
   (ix) cgt. if $x \leqslant 1$, div. if $x > 1$;
   (x) cgt. if $x \leqslant 1$, div. if $x > 1$;
   (xi) cgt. if $x \leqslant 1$, div. if $x > 1$.
4. (i) cgt. if $x < 1$, div. if $x \geqslant 1$;
   (ii) cgt. if $x < 1$, div. if $x > 1$; when $x = 1$, cgt. if $p > 1$ and div. if $p \leqslant 1$;

(iii) cgt. if $x \leqslant 1$, div. if $x > 1$;
(iv) cgt. if $x < 1$, div. if $x \geqslant 1$;
(v) cgt. if $x < 1$, div. if $x \geqslant 1$;
(vi) cgt. if $x < 1$, div. if $x \geqslant 1$.
5. (i) cgt. if $x < 1$, div. if $x \geqslant 1$;
(ii) cgt. if $x > 1$, div. if $x \leqslant 1$.
6. cgt. if $a < 1$, div. if $a \geqslant 1$.
7. cgt. if $x \leqslant 1$ or $x \geqslant -1$, div. if $x > 1$ or $x < -1$.
8. div. 9. (i) cgt.; (ii) cgt.

Problem Set 5

1. (i) cgt. if $x \leqslant 1$, div. if $x > 1$;
(ii) cgt. if $x \leqslant 1$, div. if $x > 1$;
(iii) cgt. if $x \leqslant 1$, div. if $x > 1$.
2. (i) cgt. if $x \leqslant 1$ or $x \geqslant -1$, div. if $x > 1$ or $x < -1$;
(ii) cgt. if $x \leqslant 1$ or $x \geqslant -1$, div. if $x > 1$ or $x < -1$.
3. (i) cgt. if $x \geqslant \frac{1}{e}$, div. if $x < \frac{1}{e}$;
(ii) cgt. if $x < e$, div. if $x \geqslant e$.
4. (i) cgt. if $x < 1$, div. if $x \geqslant 1$;
(ii) cgt. if $x < 1$, div. if $x > 1$, when $x = 1$ cgt. if $\beta > 2\alpha$ and div. if $\beta \leqslant 2\alpha$.
5. (i) div.; (ii) cgt. if $b - a > 1$, div. if $b - a \leqslant 1$;
(iii) cgt. if $\gamma + \delta - \alpha - \beta > 1$, div. if $\gamma + \delta - \alpha - \beta \leqslant 1$.
6. (i) div.; (ii) cgt. 7. (i) div.; (ii) div.; (iii) cgt.
8. cgt. if $x < \frac{1}{e}$, div. if $x \geqslant \frac{1}{e}$. 9. div.

Problem Set 6

2. cgt. 4. cgt. 5. (i) cgt.; (ii) cgt. (iii) cgt.

7. (a) (i) cgt. if $|x| < 1$, cgt. if $|x| = 1$, div. if $x > 1$, not cgt. if $x < -1$;

(ii) cgt. if $|x| < 1$, div. if $x = 1$, cgt. if $x = -1$, div. if $x > 1$, not cgt. if $x < -1$. 7. (b) absolutely cgt.

8. (i) cgt. if $|x| < 1$, conditionally cgt. if $x = 1$, div. if $x = -1$, div. if $x < -1$, not cgt. if $x > 1$; (ii) cgt.

9. cgt. if $x > 0$, oscillates finitely if $x = 0$, oscillates infinitely if $x < 0$.

Problem Set 7

1. does not exist. **2.** exists. **3.** (ii) Discontinuous at $x=0$. **4.** (i) continuous at $x = 1, 2$ but discontinuous at $x = 0$. (ii) Discontinuous at $x=0, \frac{1}{2}$. **7.** (i) discontinuous at $x = 0, 1$. **8.** continuous. **10.** (i) no; $f(0) = 7$. (ii) $f(x)=4$ at $x=2$.

11. (i) continuous but not differentiable;
(ii) continuous but not differentiable. **12.** 0.

14. (i) differentiable at $x = 0$;
(ii) differentiable at $x = \dfrac{\pi}{2}$.

15. (i) continuous for all x; (ii) continuous for all x.

17. $\{0\}$.

HINTS

Hints to Problem Set 1

1. (i) Let x belong to the set R^- of negative real numbers.

Then $x < 0 \;\forall\; x \in R^-$. So 0 is an upper bound. If possible let g be a lower bound of R^-. So $x \geqslant g \;\forall\; x \in R^-$.

As $-1 \in R^- \quad -1 \geqslant g$. Also $0 > -2$.

$\therefore \quad -1 + 0 > g - 2$ or $0 > g - 1$. So $g - 1 \in R^-$.

But $S \geqslant g \;\forall\; x \in R^{-1}$. So $g - 1 \geqslant g$ i.e. $-1 \geqslant 0$ which is absurd. So R^- is not bounded below.

(ii) Let l be an upper bound.

Then $x \leqslant l \;\forall\; x \in R$. As $l + 1 \in R$ we get $l + 1 \leqslant l$ i.e. $1 \leqslant 0$ (absurd). Hence R is not bounded above.

Similarly if g is a lower bound $g - 1 \geqslant g$ (absurd).

So R is not bounded below. So R is unbounded.

2. (i) $x = n^2 + \dfrac{1}{n^2} + 1 = 3 + n^2 + \dfrac{1}{n^2} - 2$

$$= 3 + \left(n - \frac{1}{n}\right)^2.$$

$\therefore \quad x \geqslant 3 \;\forall\; x \in A$, (the set). So 3 is a lower bound.

$\therefore$ By continuum property, A has the greatest lower bound g. So $x \geqslant g \geqslant 3$. Let $g = 3 + \epsilon$. So $x \geqslant 3 + \epsilon$

or $$3 + \left(n - \frac{1}{n}\right)^2 \geqslant 3 + \epsilon \;\forall\; n \in N.$$

Take $n = 1$. Then $3 \geqslant 3 + \epsilon$ or $0 \geqslant \epsilon$.
But $0 > \epsilon$ implies $g = 3 + \epsilon < 3 + 0 = 3$.
This contradicts the fact $g \geqslant 3$. So $\epsilon = 0$. So $g = 3$.

(ii) $x = 3 - n + \dfrac{1}{n} \leqslant 3 \;\forall\; x \in A$, (the set) as $n \in N$.

$\therefore$ 3 is an upper bound.
By continuum property A has the least upper bound l.
So $x \leqslant l \leqslant 3$. Let $l = 3 - \epsilon$. So $x \leqslant 3 - \epsilon$;

or $$3 - n + \frac{1}{n} \leqslant 3 - \epsilon \;\forall\; n \in N.$$

Take $n = 1$. Then $3 \leqslant 3 - \epsilon$; $\therefore\ \epsilon \leqslant 0$.
But $\epsilon < 0$ implies $l = 3 - \epsilon > 3 - 0 = 3$.
This contradicts the fact $l \leqslant 3$. So $\epsilon = 0$.
Hence $l = 3 - \epsilon = 3 - 0 = 3$.

(iii) $x > 0$. So 0 is a lower bound.
By continuum property the set A has the greatest lower bound g. So $x \geqslant g \;\forall\; x \in A$.
Now $g \not< 0$ because 0 is a lower bound.
If $g > 0$ we can find $\epsilon > 0$ such that $g - \epsilon > 0$.
So $g - \epsilon$ belongs to A. So $g - \epsilon \geqslant g$
i.e. $0 \geqslant \epsilon$ which contradicts the fact $\epsilon > 0$.
Hence only possibility is $g = 0$.

(iv) [Note. Read $F \leqslant R$ as $F \subseteq R$.]
As x is a lower bound, by continuum property of R (whose subset is F) the greatest lower bound i.e. sup $F = \xi$ exists.
So $\xi \geqslant x$.
Now $a \geqslant \xi$ for all $a \in F$. As $x \in F$ we get $x \geqslant \xi$.
$\therefore$ $\xi \geqslant x$ as well as $x \geqslant \xi$. So $\xi = x$.
As y is an upper bound, by continuum property the least upper bound i.e. inf $F = \eta$ exists. So $\eta \leqslant y$.
As $\eta \geqslant a$ for all $a \in F$ and $y \in F$, $\eta \geqslant y$.
$\therefore$ $\eta \leqslant y$ as well as $\eta \geqslant y$. So $\eta = y$.

3. [Note. Read $n \leqslant x \leqslant n + 1$ as $n \leqslant x < n + 1$]
Let $x \in R$. We have to prove that there exists an integer n such that $n \leqslant x < n + 1$.

Take the set $A = \{m \mid m < x.\ m \in Z\}$. This set is bounded above by x. So it has the suprenuum $n \in Z$.

$\therefore\ x \geqslant n$ and $n > x - 1$. Thus $n + 1 > x \geqslant n$.]

Hints to Problem Set 2

1. $$\left| u_n - \frac{1}{2} \right| < \epsilon \Rightarrow \left| \frac{-3}{4n^2 + 10} \right| < \epsilon$$

$$\Rightarrow \frac{3}{4n^2 + 10} < \epsilon \Rightarrow n > \sqrt{\frac{3}{4\epsilon} - \frac{5}{2}}$$

$\therefore\ m =$ integer greater than $\sqrt{\dfrac{3}{4\epsilon} - \dfrac{5}{2}}$.

2. $| u_n | = \left| \dfrac{\sin\frac{1}{2}n\pi}{n} \right| \leqslant \dfrac{1}{n}$ as $\left| \sin\frac{1}{2}n\pi \right| \leqslant 1$.

$\therefore\ | u_n | < \epsilon$ whenever $n > \dfrac{1}{\epsilon}$

$\therefore$ Given $\epsilon > 0$ there exists a $+ve$ integer $m =$ integer greater than $\dfrac{1}{\epsilon}$ such that $| u_n | < \epsilon$ whenever $n \geqslant m$;

So $u_n \to 0$.

3. $$u_n > \frac{1}{n+n} + \frac{1}{n+n} + \ldots + \frac{1}{n+n} = \frac{n}{n+n} = \frac{1}{2}$$

$$u_n < \frac{1}{n+1} + \frac{1}{n+1} + \ldots + \frac{1}{n+1} = \frac{n}{n+1} < 1;$$

$$\therefore\ \tfrac{1}{2} < u_n < 1 \ \forall\ n \in N.$$

$$u_{n+1} - u_n = \left(\frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{2n+2}\right) - \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right)$$

$$= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{1}{2(n+1)(2n+1)} > 0$$

$\therefore\ \{u_n\}$ is bounded and *m.i.* and so it tends to a limit.

4. Take $b_n = \dfrac{1}{n} \to 0.$

Then $a_n = \dfrac{1}{n}\left(1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots. + \dfrac{1}{n}\right)$

$$= \frac{b_1 + b_2 + \ldots. + b_n}{n}.$$

By Cauchy's first limit, $a_n \to 0$ because $b_n \to 0$.

5. $\lim\limits_{n\to\infty} a_n > 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8})$

$+ \ldots.$to ∞

i.e. $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots.$to ∞.

Let $b_n = 1 + \frac{1}{2} + \frac{1}{2} + \ldots.$to n terms

$$= 1 + \frac{n-1}{2} \to \infty \text{ as } n \to \infty.$$

$\therefore \lim\limits_{n\to\infty} a_n > \lim\limits_{n\to\infty} b_n = \infty;\quad \therefore a_n \to \infty.$

6. $a_n = 1 + \dfrac{1}{2^{n-1}},\ a_{n+1} = 1 + \dfrac{1}{2^n}$

$\therefore\ a_n - a_{n+1} = \dfrac{1}{2^{n-1}} > 0;$

$\therefore$ $\{a_n\}$ is monotonic decreasing.

Also $a_n = 1 + \dfrac{1}{2^{n-1}} > 1.$

So $\{a_n\}$ is *m.d.* and bounded below.

$\therefore$ $\{a_n\}$ is convergent to l (say).

Also $2a_n = 2 + \dfrac{1}{2^n} = 1 + a_{n+1}$. So when $n \to \infty$,

$$2l = 1 + l;\quad \therefore\ l = 1.$$

7. $u_{n+1} - u_n = \dfrac{1.3.5\ldots.(2n-1)}{3.6.9\ldots.3n}\left(\dfrac{2n+1}{3n+3} - 1\right) < 0$

and $u_n > 0$.

$\therefore$ $\{u_n\}$ is *m.d.* and bounded below by 0.

So it is convergent to a limit greater than or equal to 0.

Again $u_n < \dfrac{3.6.9....3n}{3.5.7....(2n+1)}$

($\because$ numerator in u_n has been increased and denominators has been decreased)

$$\therefore \quad u_n.u_n < \frac{1.3.5....(2n-1)}{3.6.9....3n} \times \frac{3.6.9....3n}{3.5.7....(2n+1)}$$

$$= \frac{1}{2n+1}.$$

$$\therefore \quad u_n < \frac{1}{\sqrt{2n+1}} \to 0. \text{ So } u_n \to 0.$$

8. (i) See worked out problem 8. Take $k = 1$.

(ii) See worked out problem 8. Take $k = 5$.

(iii) $a_1 = \sqrt{7}$, $a_2 = \sqrt{7+\sqrt{7}} = \sqrt{7+a_1}$,

$$a_3 = \sqrt{7+\sqrt{7+\sqrt{7}}} = \sqrt{7+a_2}, \ldots,$$

$\therefore \quad a_{n+1} = \sqrt{7+a_n}$. See worked out problem 8. Take $k = 7$.

(iv) As above $a_{n+1} = \sqrt{2+a_n}$. Take $k = 2$ in worked out problem 8.

9. (i) $a_{n+1}^2 - a_n^2 = 2a_n - 2a_{n-1}$;

$\therefore \quad a_{n+1} >$ or $< a_n$ according as $a_n >$ or $< a_{n-1}$.

$\therefore \quad \{a_n\}$ is monotonic. Also $a_2 > a_1$. So it is *m.i.*

Also $a_n^2 = 2a_{n-1} < 2a_n$;

$\therefore \quad a_n < 2$ for a_n is positive as $a^2_{n+1} = 2a_n$.

$\therefore \quad \{a_n\}$ is *m.i.* and bounded above.

So it is convergent to the least upper bound l (say).

$\therefore$ From $a_n^2 = 2a_{n-1}$ we get when $n \to \infty$, $l^2 = 2l$;

$$\therefore \quad l = 2.$$

(ii) $a_1 = \sqrt{2}$, $a_2 = \sqrt{2a_1}$ i.e. $a_2^2 = 2a_1$

$a_3 = \sqrt{\{2\sqrt{(2\sqrt{2})}\}} = \sqrt{2a_2}$ i.e. $a_3^2 = 2a_2, \ldots;$

Thus $a_n^2 = 2a_{n-1}$.

See the above sum (i).

10. $S_n = \dfrac{2S_{n-1}S_{n-2}}{S_{n-1} + S_{n-2}}$.

means S_n is the harmonic mean between S_{n-1}, S_{n-2}.

By question, $S_1 < S_2 \Rightarrow \dfrac{1}{S_1} > \dfrac{1}{S_2}$.

As S_3 is H.M. between S_1, S_2 we get $\dfrac{1}{S_3}$ is A.M. between

$$\frac{1}{S_1}, \frac{1}{S_2}.$$

$$\therefore \quad \frac{1}{S_1} > \frac{1}{S_3} > \frac{1}{S_2} \Rightarrow S_1 < S_3 < S_2 \quad \ldots \quad (1)$$

S_4 is the H.M. between S_2 and S_3.

$\because \quad S_3 < S_2$, from (1); $\therefore \quad S_3 < S_4 < S_2$. .. (2)

From (1) and (2), we get

$$S_1 < S_3 < S_4 < S_2. \quad \ldots \quad (3)$$

S_5 is the H.M. between S_3 and S_4.

$\because \quad S_3 < S_4$, from (2); $\therefore \quad S_3 < S_5 < S_4$. .. (4)

From (3) and (4), we get

$$S_1 < S_3 < S_5 < S_4 < S_2.$$

Repeating this process we get

$$S_1 < S_3 < S_5 < S_7 < \ldots < S_6 < S_4 < S_2.$$

Hence the sequence $\{S_{2n+1}\}$ is monotonic increasing and bounded above by S_2.

Again the sequence $\{S_{2n}\}$ is monotonic decreasing and bounded below by S_1.

Hence both the sequences $\{S_{2n+1}\}$ and $\{S_{2n}\}$ must be convergent.

Let $\lim\limits_{n\to\infty} S_{2n} = l \Rightarrow \lim\limits_{n\to\infty} S_{2n-2} = l$; and $\lim\limits_{n\to\infty} S_{2n-1} = m$.

$\because \quad S_{2n} = \dfrac{2S_{2n-1}S_{2n-2}}{S_{2n-1} + S_{2n-2}}$, making $n \to \infty$

$$l = \frac{2ml}{m+l}; \quad \therefore \quad l = m.$$

Hence both the sequences converge to the same limit.

Again, $\because \quad S_n = \dfrac{2S_{n-1}S_{n-2}}{S_{n-1} + S_{n-2}}$

$$\therefore \quad \frac{2}{S_n} = \frac{1}{S_{n-2}} + \frac{1}{S_{n-1}} \text{ for all } n.$$

$$\begin{aligned}
\therefore \quad \frac{2}{S_n} + \frac{1}{S_{n-1}} &= \frac{1}{S_{n-2}} + \frac{2}{S_{n-1}} \\
&= \frac{1}{S_{n-2}} + \left(\frac{1}{S_{n-2}} + \frac{1}{S_{n-3}}\right) = \frac{2}{S_{n-2}} + \frac{1}{S_{n-3}} \\
&= \left(\frac{1}{S_{n-3}} + \frac{1}{S_{n-4}}\right) + \frac{1}{S_{n-3}} = \frac{2}{S_{n-3}} + \frac{1}{S_{n-4}} \\
&= \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\ldots \\
&= \frac{2}{S_2} + \frac{1}{S_1}.
\end{aligned}$$

Making $n \to \infty$, $\dfrac{2}{l} + \dfrac{1}{l} = \dfrac{2}{S_2} + \dfrac{1}{S_1}$

$$\frac{3}{l} = \frac{2S_1 + S_2}{S_1S_2}; \quad \therefore \quad l = \frac{3S_1S_2}{2S_1 + S_2}.$$

11. $\because \quad a_{n+1} = \frac{1}{2}(a_n + b_n)$

a_{n+1} is the A.M. between a_n and b_n.

Again from $b_{n+1} = \dfrac{2a_nb_n}{a_n + b_n}$,

b_{n+1} is the H.M. between a_n and b_n.

As A.M. > H.M. and a_2 is the A.M. and b_2 is the H.M. between a_1 and b_1.

$$\therefore \quad a_2 > b_2 \qquad \ldots \quad (1)$$

CASE I. Let $a_1 > b_1$.

$\because \quad a_2$ is the A.M. between a_1 and b_1,

$$\therefore \quad a_1 > a_2 > b_1. \qquad \ldots \quad (2)$$

Again $b_2 = \dfrac{2a_1b_1}{a_1 + b_1} = \dfrac{2a_1b_1}{2a_2}$; $\Rightarrow \quad \dfrac{b_2}{b_1} = \dfrac{a_1}{a_2} > 1$;

$$\Rightarrow \quad b_2 > b_1. \qquad \ldots \quad (3)$$

$\therefore$ From (1), (2) and (3), we get

$$a_1 > a_2 > b_2 > b_1.$$

Repeating this process, we get

$$a_1 > a_2 > a_3 > \ldots . > b_3 > b_2 > b_1.$$

Hence the sequence $\{a_n\}$ and $\{b_n\}$ are monotonic and bounded sequences. Hence both are convergent.

CASE II. Let $b_1 > a_1$.

As in case I, we can prove that both the sequences $\{a_n\}$ and $\{b_n\}$ are convergent.

In both the cases they must tend to a limit.

Let $\lim\limits_{n\to\infty} a_n = l = \lim\limits_{n\to\infty} a_{n+1}$ and $\lim\limits_{n\to\infty} b_n = m = \lim\limits_{n\to\infty} b_{n+1}$;

as $$a_{n+1} = \frac{a_n + b_n}{2}$$

$$\therefore \lim_{n\to\infty} a_{n+1} = \tfrac{1}{2}\left(\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n\right)$$

$$\therefore\ l = \tfrac{1}{2}(l+m);\quad \therefore\ 2l = l + m;\quad \therefore\ l = m.$$

Hence both the sequences tend to the same limit.

Also $$a_{n+1}.b_{n+1} = \frac{a_n + b_n}{2} . \frac{2a_nb_n}{a_n + b_n} = a_nb_n = a_{n-1}b_{n-1}$$

$$= \ldots . = a_1b_1$$

$$\therefore\ \lim_{n\to\infty} a_{n+1}. \lim_{n\to\infty} b_{n+1} = a_1b_1.$$

$$\Rightarrow\ l^2 = a_1b_1;\quad \therefore\ l = \sqrt{a_1b_1}.$$

This is the required limit.

12. The given limit

$$= \lim_{n\to\infty} \frac{(\lfloor n)^{\frac{1}{n}}}{n} = \lim_{n\to\infty} \frac{(1.2.3\ldots .n)^{\frac{1}{n}}}{n}$$

$$= \lim_{n\to\infty} \left[\frac{(1.2.3\ldots .n)^{\frac{1}{n}}}{\{(n+1)^n\}^{\frac{1}{n}}} . \frac{n+1}{n} \right]$$

$$= \lim_{n\to\infty} \left\{ \left[\frac{1}{2} . \left(\frac{2}{3}\right)^2 . \left(\frac{3}{4}\right)^3 \ldots \left(\frac{n}{n+1}\right)^n \right]^{\frac{1}{n}} . \frac{n+1}{n} \right\}$$

$$= \lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^n \cdot \frac{n+1}{n}\right]$$

$$= \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = \frac{1}{e}.$$

13. As $a_n > 0 \ \forall\ n \in N$, $k > 0$ and $a_1 > 0$.

Also $0 < a_n = \dfrac{k}{1+a_{n-1}} < k$,

i.e. $\{a_n\}$ is a bounded sequence.

$$a_n = \frac{k}{1+a_{n-1}} = \frac{k}{1+\dfrac{k}{1+a_{n-2}}};$$

$$\therefore \quad a_n - a_{n-2} = \frac{k - a_{n-2} - a_{n-2}^2}{1+k+a_{n-2}}$$

$$a_n - a_{n-1} = \frac{k(1+a_{n-2})}{1+k+a_{n-2}} - \frac{k}{1+a_{n-2}}$$

$$= \frac{-k(k - a_{n-2} - a_{n-2}^2)}{(1+k+a_{n-2})(1+a_{n-2})}$$

$$= \frac{-k}{1+a_{n-2}}.(a_n - a_{n-2})$$

$\therefore \quad a_n > a_{n-1}$ according as $a_n < a_{n-2}$.

$\therefore \quad$ If $a_1 > a_2$ then $a_1 > a_3 > a_5 > \ldots > a_6 > a_4 > a_2$

If $a_1 < a_2$ then $a_1 < a_3 < a_5 < \ldots < a_6 < a_4 < a_2$

Thus $\{a_{2n}\}$ and $\{a_{2n+1}\}$ are both bounded and monotonic, one increasing and the other decreasing. Hence they are convergent.

Let $a_{2n} \to l$, $a_{2n+1} \to l'$. Then from $a_n = \dfrac{k}{1+a_{n-1}}$ we get

$l = \dfrac{k}{1+l'}$ and $l' = \dfrac{k}{1+l}$ according as n is even or odd;

i.e. $l + ll' = k = l' + ll'$; $\therefore \quad l = l'$.

Thus both the sequences $\{a_{2n}\}$ and $\{a_{2n+1}\}$ tend to the same limit.

$$\therefore\ l = \frac{k}{1+l}; \quad \text{or} \quad l^2 + l - k = 0.$$

But l is positive as $0 < a_n < k$. So $\lim\limits_{n\to\infty} a_n =$ positive root l.

14. Let $a_n = a + \epsilon_n$ where $\epsilon_n \to 0$. Then

$$\frac{a_1b_n + a_2b_{n-1} + \ldots + a_nb_1}{n}$$

$$= a.\frac{b_1 + b_2 + \ldots + b_n}{n} + \frac{\epsilon_1b_1 + \ldots + \epsilon_nb_n}{n}.$$

As $b_n \to b$, by Cauchy's first limit

$$\frac{b_1 + b_2 + \ldots + b_n}{n} \to b.$$

As $b_n \to b$, $\{b_n\}$ is bounded and so $|\, b_n \,| < k$ for all n.

$$\therefore \left| \frac{\epsilon_1b_1 + \epsilon_2b_2 + \ldots + \epsilon_nb_n}{n} \right|$$

$$< \frac{|\, \epsilon_1 \,| + |\, \epsilon_2 \,| + \ldots + |\, \epsilon_n \,|}{n} k \to 0$$

because $|\, \epsilon_n \,| \to 0$ (by Cauchy's first limit).

Hence $\dfrac{a_1b_n + \ldots + a_nb_1}{n} \to a\,.\,b + 0.$

15. As $a_1 > 0$, from $a_{n+1} = \dfrac{3 + 2a_n}{2 + a_n}$ we get

$$a_n > 0 \ \forall\ n \in N \text{ and } a_{n+1} < \frac{4 + 2a_n}{2 + a_n} = 2,$$

$$a_{n+1} > \frac{3 + 2a_n}{3 + 2a_n} = 1. \quad \text{So } 1 < a_n < 2 \ \forall\ n \in N.$$

$$\text{Now } a_{n+1} - a_n = \frac{3 + 2a_n}{2 + a_n} - \frac{3 + 2a_{n-1}}{2 + a_{n-1}}$$

$$= \frac{a_{n-1} - a_n}{(2 + a_n)(2 + a_{n-1})}.$$

Let $a_1 > a_2$; then $a_3 - a_2 = \dfrac{a_1 - a_2}{(2 + a_2)(2 + a_1)} > 0$

i.e. $$a_3 > a_2$$

$\because \quad a_3 > a_2$ then $a_4 - a_3 = \dfrac{a_2 - a_3}{(2 + a_3)(2 + a_2)} < 0$

i.e. $$a_4 < a_3 \text{ etc.}$$

$\therefore \quad a_2 < a_4 < a_6 < \ldots . < a_5 < a_3 < a_1$

Similarly, if $a_1 < a_2$ we get

$$a_2 > a_4 > a_6 > \ldots . > a_3 > a_1.$$

$\therefore \quad \{a_{2n+1}\}$ and $\{a_{2n}\}$ are both monotonic and bounded.

Let $a_{2n+1} \to l$, $a_{2n} \to l'$.

Then as $n \to \infty$, from $a_{n+1} = \dfrac{3 + 2a_n}{2 + a_n}$,

$$l = \frac{3 + 2l'}{2 + l'} \text{ and } l' = \frac{3 + 2l}{2 + l}; \quad \therefore \quad l = l'.$$

$\therefore \quad$ When $n \to \infty$, from $a_{n+1} = \dfrac{3 + 2a_n}{2 + a_n}$ we get

$$l = \frac{3 + 2l}{2 + l} \quad \text{i.e.} \quad l = \sqrt{3}.$$

16. (i) As $l > 1$, choose $\epsilon > 0$ such that $l - \epsilon > 1$.

Now $\lim\limits_{n \to \infty} \dfrac{a_{n+1}}{a_n} = l \Rightarrow \left| \dfrac{a_{n+1}}{a_n} - l \right| < \epsilon;$

$\therefore \quad \dfrac{a_{n+1}}{a_n} > l - \epsilon > 1$ for $n \geqslant m$.

$$\therefore \quad \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \ldots . \frac{a_{m+1}}{a_m} > (l - \epsilon)^{n-m}$$

$$\therefore \quad \frac{a_n}{a_m} > (l - \epsilon)^{n-m} \Rightarrow |\, a_n \,| > |\, a_m \,| \, (l - \epsilon)^{n-m} \to \infty$$

$\therefore \quad \{a_n\}$ diverges to $+ \infty$ (or $- \infty$ as terms will be of the same sign.)

(ii) $|\, l \,| < 1 \Rightarrow$ we can choose $\epsilon > 0$ such that

$$|\, l \,| + \epsilon < 1.$$

Now $\left| \frac{a_{n+1}}{a_n} - l \right| < \epsilon$

$$\Rightarrow \left| \frac{a_{n+1}}{a_n} \right| - | l | < \left| \frac{a_{n+1}}{a_n} - l \right| < \epsilon \text{ for } n \geqslant m.$$

$\therefore \left| \frac{a_{n+1}}{a_n} \right| < | l | + \epsilon < 1$. So as above we get,

$$| a_{n+1} | < (| l | + \epsilon)^{n-m} \, | a_m | \to 0 \text{ as } n \to \infty.$$

17. Let $a_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}$

$$\leqslant \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+1}} + \dots + \frac{1}{\sqrt{n^2+1}}$$

$\therefore \quad a_n \frac{\leqslant n}{\sqrt{n^2+1}}$. Also $a_n \geqslant \frac{1}{\sqrt{n^2+n}} + \frac{1}{\sqrt{n^2+n}} + \dots$

$$+ \frac{1}{\sqrt{n^2+n}} = \frac{n}{\sqrt{n^2+n}}$$

$$\therefore \quad \frac{n}{\sqrt{n^2+n}} \leqslant a_n \leqslant \frac{n}{\sqrt{n^2+1}}$$

$\therefore$ When $n \to \infty$, $1 \leqslant \lim a_n \leqslant 1$. $\quad \therefore \lim a_n = 1$.

Hints to Problem Set 3

1. (i) Here $u_n = \frac{5n-3}{2n^3-1}$.

Take $v_n = \frac{1}{n^2}$; $\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{5}{2}$.

But $\sum v_n$ converges and so $\sum u_n$ converges.

(ii) Here $u_n = \frac{3n+1}{n^3+1}$. Take $v_n = \frac{1}{n^2}$; $\lim_{n\to\infty} \frac{u_n}{v_n} = 3$.

But $\sum v_n$ converges and so $\sum u_n$ converges.

(iii) Here $u_n = \frac{2n-1}{2n^2-2n+1}$.

Take $v_n = \frac{1}{n}$; $\lim_{n\to\infty} \frac{u_n}{v_n} = 1$.

But $\sum v_n$ is divergent and so $\sum u_n$ diverges.

(iv) Here $u_n = \dfrac{2n^2+1}{3n^3+5n^2+6}$.

Take $v_n = \dfrac{1}{n}$; $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \dfrac{2}{3}$.

But $\sum v_n$ is divergent and so $\sum u_n$ is divergent.

2. (i) Here $u_n = \sqrt{\dfrac{n}{(n+1)^3}}$.

Take $v_n = \dfrac{1}{n}$; $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = 1$.

But $\sum v_n$ is divergent and so $\sum u_n$ diverges.

(ii) Here $u_n = \dfrac{2n-1}{n(n+1)(n+2)}$.

Take $v_n = \dfrac{1}{n^2}$; $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = 2$.

But $\sum v_n$ converges. So $\sum u_n$ is convergent.

(iii) $u_n = \dfrac{n+1}{n^2}$. Take $v_n = \dfrac{1}{n}$; $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = 1$.

But $\sum v_n$ diverges. So $\sum u_n$ also diverges.

(iv) $u_n = \dfrac{1}{(2n-1)^2}$. Take $v_n = \dfrac{1}{n^2}$; $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \dfrac{1}{4}$.

But $\sum v_n$ is convergent. So $\sum u_n$ is also convergent.

(v) $u_n = \dfrac{\sqrt{n}}{n+\sqrt{n}}$ Take $v_n = \dfrac{1}{\sqrt{n}}$; $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = 1$.

But $\sum v_n$ is divergent. So $\sum u_n$ is also divergent.

(vi) $u_n = \dfrac{1}{n(n+1)(n+1)}$.

Take $v_n = \dfrac{1}{n^3}$; $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = 1$.

But $\sum v_n$ is convergent. So $\sum u_n$ is also convergent.

3. (i) $u_n = \dfrac{\sqrt{n}}{n^2+1}$. Take $v_n = \dfrac{1}{n^{\frac{3}{2}}}$; $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = 1$.

But $\sum v_n$ is convergent. So $\sum u_n$ is also convergent.

(ii) $u_n = \dfrac{1}{\sqrt{1+n^2}+n}$. Take $v_n = \dfrac{1}{n}$; $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \dfrac{1}{2}$.

But $\sum v_n$ is divergent. So $\sum u_n$ is also divergent.

(iii) $u_n = \dfrac{\sqrt{n+1}+\sqrt{n}}{n^2}$. Take $v_n = \dfrac{1}{n^{\frac{3}{2}}}$; $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = 2$

But $\sum v_n$ is convergent. So $\sum u_n$ is also convergent.

(iv) $u_n = \left(\dfrac{n^2+2}{2n^3+15}\right)^{\frac{1}{3}}$. Take $v_n = \dfrac{1}{n^{\frac{1}{3}}}$; $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \dfrac{1}{\sqrt[3]{2}}$.

But $\sum v_n$ is divergent. So $\sum u_n$ is also divergent.

4. (i) $u_n = \sqrt{n^2+1} - \sqrt{n^2-1} = \dfrac{2}{\sqrt{n^2+1}+\sqrt{n^2-1}}$.

Take $v_n = \dfrac{1}{n}$; $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = 1$.

But $\sum v_n$ is divergent; so $\sum u_n$ is divergent.

(ii) $u_n = \sqrt{n^3+1} - \sqrt{n^3} = \dfrac{1}{\sqrt{n^3+1}+\sqrt{n^3}}$.

Take $v_n = \dfrac{1}{n^{\frac{3}{2}}}$; $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \dfrac{1}{2}$.

But $\sum v_n$ is convergent; so $\sum u_n$ is convergent.

(iii) $u_n = \sqrt{n^5+1} - \sqrt{n^5} = \dfrac{1}{\sqrt{n^5+1}-\sqrt{n^5}}$.

Take $v_n = \dfrac{1}{n^{\frac{5}{2}}}$; $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \dfrac{1}{2}$.

But $\sum v_n$ is convergent; so $\sum u_n$ is convergent.

(iv) $u_n = \sqrt{n^3+1} - \sqrt{n^3-1} = \dfrac{2}{\sqrt{n^2+1}+\sqrt{n^3-1}}$.

Take $v_n = \dfrac{1}{n^{\frac{3}{2}}}$; $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = 1$.

But $\sum v_n$ is convergent; so $\sum u_n$ is convergent.

(v) $u_n = \sqrt{n^2+1} - n = \dfrac{1}{\sqrt{n^2+1}+n}$.

Take $v_n = \frac{1}{n}$; $\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{2}$.

But $\sum v_n$ is divergent; so $\sum u_n$ is divergent.

5. (i) $u_n = \sqrt{n^4+1} - n^2 = \frac{1}{\sqrt{n^4+1}+n^2}$.

Take $v_n = \frac{1}{n^2}$; $\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{2}$.

But $\sum v_n$ is convergent; so $\sum u_n$ is convergent.

(ii) $u_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$.

Take $v_n = \frac{1}{n^{\frac{1}{2}}}$; $\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{2}$.

But $\sum v_n$ is divergent; so $\sum u_n$ is divergent.

6. (i) $u_n = \frac{1}{(2n-1)^p}$. Take $v_n = \frac{1}{n^p}$; $\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{2^p}$

But $\sum v_n$ is convergent or divergent according as
$p > 1$ or $\leqslant 1$.

$\therefore$ $\sum u_n$ is convergent or divergent according as
$p > 1$ or $\leqslant 1$.

(ii) $u_n = \frac{\sqrt{n+1}-\sqrt{n}}{n^p} = \frac{1}{n^p(\sqrt{n+1}+\sqrt{n})}$.

Take $v_n = \frac{1}{n^{p+\frac{1}{2}}}$; $\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{2}$.

But $\sum v_n$ is cgt. or div. according as $p + \frac{1}{2} > 1$ or $\leqslant 1$.
So $\sum u_n$ is cgt. or div. according as $p > \frac{1}{2}$ or $\leqslant \frac{1}{2}$.

7. (i) $u_n = \frac{\cos^2 nx}{(2n+1)^2} \leqslant \frac{1}{(2n+1)^2} = V_n$;

take $v_n = \frac{1}{n^2}$; $\lim_{n\to\infty} \frac{V_n}{v_n} = \frac{1}{4}$.

But $\sum v_n$ is cgt.; so $\sum V_n$ is cgt.
As $u_n \leqslant V_n$, $\sum u_n$ is also cgt.

(ii) $u_n = \frac{1}{n} \sin \frac{1}{n}$; take $v_n = \frac{1}{n^2}$; $\lim\limits_{n\to\infty} \frac{u_n}{v_n} = 1.$

But $\sum v_n$ is cgt.; so $\sum u_n$ is cgt.

(iii) $u_n = \tan^{-1}\frac{1}{n}$; take $v_n = \frac{1}{n}$;

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\tan^{-1}\frac{1}{n}}{\frac{1}{n}} = \lim_{\theta\to\infty} \frac{\theta}{\tan\theta} = 1.$$

But $\sum v_n$ is div.; so $\sum u_n$ is div.

Hints to Problem Set 4

1. $u_n = \frac{2.4.6....(2n)}{3^2.5^2.7^2....(2n+1)^2}$, leaving the first term.

$\therefore \frac{u_{n+1}}{u_n} = \frac{2n+2}{(2n+3)^2}$; $\therefore \lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = 0 < 1.$

So $\sum u_n$ is cgt. by D'Alembert's test.

2. (i) Here $\frac{u_{n+1}}{u_n} = \frac{(n+1)^{-p}}{\underline{|n+1}} \cdot \frac{\underline{|n}}{n^{-p}}$

$$= \frac{1}{n+1}\left(1+\frac{1}{n}\right)^{-p} \to 0 < 1 \text{ as } n \to \infty.$$

$\therefore$ $\sum u_n$ is cgt. by D'Alembert's test.

(ii) Here $u_n = \frac{\underline{|n}}{n^n}$;

$$\therefore \frac{u_n}{u_{n+1}} = \frac{\underline{|n}}{n^n} \cdot \frac{(n+1)^{n+1}}{\underline{|n+1}} = \left(1+\frac{1}{n}\right)^n.$$

$\therefore \lim\limits_{n\to\infty} \frac{u_n}{u_{n+1}} = e > 1;$

$\therefore$ $\sum u_n$ is cgt. by D'Alembert's test.

3. (i) $\frac{u_n}{u_{n+1}} = \frac{2n-5}{n^3-5n} \cdot \frac{(n+1)^3 - 5(n+1)}{2(n+1)-5} \cdot \frac{1}{x}$

$$\to \frac{1}{x} \text{ as } n \to \infty.$$

$\therefore$ By D'Alembert's test, $\sum u_n$ is cgt. if $x < 1$ and divergent if $x > 1$.

When $x = 1$, $u_n = \dfrac{2n-5}{n^3 - 5n}$; take $v_n = \dfrac{1}{n^2}$.

Clearly $\dfrac{u_n}{v_n} \to 2$ and $\sum v_n$ is cgt. So $\sum u_n$ is also cgt. by comparison test.

(ii) $u_n = \dfrac{x^{n-1}}{2n-1}$; $\therefore$ $\dfrac{u_n}{u_{n+1}} = \dfrac{2n+1}{2n-1}.\dfrac{1}{x} \to \dfrac{1}{x}$.

$\therefore$ By D'Alemberts test, $\sum u_n$ is cgt. if $x < 1$
and div. if $x > 1$.

When $x = 1$, $u_n = \dfrac{1}{2n-1}$; take $v_n = \dfrac{1}{n}$.

Clearly $\dfrac{u_n}{v_n} \to \dfrac{1}{2}$ and $\sum v_n$ is div. So by comparison test $\sum u_n$ is div.

(iii) $u_n = \dfrac{2n-1}{2n^2 - 2n + 1}x^{n-1}$;

so $\dfrac{u_n}{u_{n+1}} = \dfrac{2n-1}{2n+1}.\dfrac{2n^2+2n+1}{2n^2-2n+1}.\dfrac{1}{x}$

$\therefore$ $\lim\limits_{n\to\infty} \dfrac{u_n}{u_{n+1}} = \dfrac{1}{x}$.

So by D'Alembert's test, $\sum u_n$ is cgt. if $x < 1$ and div. if $x > 1$.

When $x = 1$, $u_n = \dfrac{2n-1}{2n^2 - 2n + 1}$; take $v_n = \dfrac{1}{n}$.

Clearly $\dfrac{u_n}{v_n} \to 1$ and $\sum v_n$ is div.; so by comparison test $\sum u_n$ is also div.

(iv) $\dfrac{u_n}{u_{n+1}} = \dfrac{2n-1}{2n+1}.\dfrac{1}{x} \to \dfrac{1}{x}$.

So by D'Alembert's test, $\sum u_n$ is cgt. if $x < 1$ and div. if $x > 1$.

When $x = 1$, the series is $1 + 3 + 5 + 7 + \ldots$ to ∞, which is clearly div.

(v) $u_n = \dfrac{x^n}{n(n+1)}$; so $\dfrac{u_n}{u_{n+1}} = \dfrac{n+2}{n}.\dfrac{1}{x} \to \dfrac{1}{x}$.

So by D'Alembert's test $\sum u_n$ is cgt. if $x < 1$ and div. if $x > 1$.

When $x = 1$, $u_n = \dfrac{1}{n(n+1)}$; take $v_n = \dfrac{1}{n^2}$.

Clearly $\dfrac{u_n}{v_n} \to 1$ and $\sum v_n$ is cgt.

$\therefore$ $\sum u_n$ is cgt. by comparison test.

(vi) $\dfrac{u_n}{u_{n+1}} = \dfrac{n}{n+1} \cdot \dfrac{(n+1)^2+1}{n^2+1} \cdot \dfrac{1}{x} \to \dfrac{1}{x}$.

So by D'Alembert's test $\sum u_n$ is cgt. if $x < 1$ and div. if $x > 1$.

When $x = 1$, $u_n = \dfrac{n}{n^2+1}$; take $v_n = \dfrac{1}{n}$.

Clearly $\dfrac{u_n}{v_n} \to 1$ and $\sum v_n$ is div.

$\therefore$ By comparison test $\sum u_n$ is also div.

(vii) $\dfrac{u_n}{u_{n+1}} = \dfrac{5n-3}{5n+2} \cdot \dfrac{2n^3+6n^2+6n+1}{2n^3-1} \cdot \dfrac{1}{x} \to \dfrac{1}{x}$.

So by D'Alembert's test $\sum u_n$ is cgt. if $x < 1$ and div. if $x > 1$.

When $x = 1$, $u_n = \dfrac{5n-3}{2n^3-1}$; take $v_n = \dfrac{1}{n^2}$.

Clearly $\dfrac{u_n}{v_n} \to \dfrac{5}{2}$ and $\sum v_n$ is cgt.

So by comparison test $\sum u_n$ is cgt.

(viii) $u_n = \dfrac{2n}{n^2+1} x^{n-1}$;

so $\dfrac{u_n}{u_{n+1}} = \dfrac{n}{n+1} \cdot \dfrac{(n+1)^2+1}{n^2+1} \cdot \dfrac{1}{x} \to \dfrac{1}{x}$.

So by D'Alembert's test $\sum u_n$ is cgt. if $x < 1$ and div. if $x > 1$.

When $x = 1$, $u_n = \dfrac{2n}{n^2+1}$; take $v_n = \dfrac{1}{n}$.

Clearly $\frac{u_n}{v_n} \to 2$ and $\sum v_n$ is div. So by comparison test $\sum u_n$ is div.

(ix) $u_n = \frac{x^{n-1}}{(n+1)\sqrt{n}}$ and

$$\frac{u_n}{u_{n+1}} = \frac{(n+2)\sqrt{n+1}}{(n+1)\sqrt{n}} \cdot \frac{1}{x} \to \frac{1}{x}.$$

So by D'Alembert's test $\sum u_n$ is cgt. if $x < 1$ and div. if $x > 1$.

When $x = 1$, $u_n = \frac{1}{(n+1)\sqrt{n}}$; take $v_n = \frac{1}{n^{\frac{3}{2}}}$.

Clearly $\frac{u_n}{v_n} \to 1$ and $\sum v_n$ is cgt. so by comparison test $\sum u_n$ is cgt.

(x) $u_n = \frac{2n-1}{(n-1)^3+1} x^{n-1}$;

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{2n-1}{2n+1} \cdot \frac{n^3+1}{(n-1)^3+1} \cdot \frac{1}{x} \to \frac{1}{x}.$$

So by D'Alembert's test $\sum u_n$ is cgt. if $x < 1$ and div. if $x > 1$.

When $x = 1$, $u_n = \frac{2n-1}{(n-1)^3+1}$; take $v_n = \frac{1}{n^2}$.

Clearly $\frac{u_n}{v_n} \to 2$ and $\sum v_n$ is cgt. So by comparison test, $\sum u_n$ is cgt.

(xi) Leaving the first term, $u_n = \frac{x^n}{n^2+1}$;

$$\therefore \quad \frac{u_n}{u_{n+1}} = \frac{(n+1)^2+1}{n^2+1} \cdot \frac{1}{x} \to \frac{1}{x}.$$

By D'Alembert's test, $\sum u_n$ is cgt. if $x < 1$ and div. if $x > 1$.

If $x = 1$, $u_n = \frac{1}{n^2+1}$; take $v_n = \frac{1}{n^2}$.

Clearly $\frac{u_n}{v_n} \to 1$ and Σv_n is cgt. So by comparison test Σu_n is cgt.

4. (i) $\frac{u_n}{u_{n+1}} = \sqrt{\frac{n^2+2n+2}{n^2+1}} \cdot \frac{1}{x} \to \frac{1}{x}$.

So by D'Alembert's test, Σu_n is cgt. if $x < 1$ and div. if $x > 1$.

When $x = 1$, $u_n = \frac{1}{\sqrt{n^2+1}}$; take $v_n = \frac{1}{n}$.

Clearly $\frac{u_n}{v_n} \to 1$ and Σv_n is div. So by comparison test Σu_n is div.

(ii) $\frac{u_n}{u_{n+1}} = \left(\frac{2n+3}{2n+1}\right)^p \cdot \frac{1}{x} \to \frac{1}{x}$.

So by D'Alembert's test, Σu_n is cgt. if $x < 1$ and div. if $x > 1$.

When $x = 1$, $u_n = \frac{1}{(2n+1)^p}$; take $v_n = \frac{1}{n^p}$.

Clearly $\frac{u_n}{v_n} \to \frac{1}{2^p}$ and Σv_n is cgt. if $p > 1$ and div. If $p \leqslant 1$.

So by comparison test Σu_n is cgt. if $p > 1$ and div. if $p \leqslant 1$.

(iii) $\frac{u_n}{u_{n+1}} = \frac{2(n+1)^2+3}{2n^2+3} \cdot \frac{1}{x} \to \frac{1}{x}$.

So by D'Alembert's test, Σu_n is cgt. if $x < 1$ and div. if $x > 1$.

When $x = 1$, $u_n = \frac{1}{2n^2+3}$; take $v_n = \frac{1}{n^2}$.

Clearly $\frac{u_n}{v_n} \to \frac{1}{2}$ and Σv_n is cgt. so by comparison test Σu_n is cgt.

(iv) $\frac{u_n}{u_{n+1}} = \frac{(n+1)\{(n+1)^2+1\}}{(n^2+1)(n+2)} \cdot \frac{1}{x} \to \frac{1}{x}$.

So by D'Alembert's test, $\sum u_n$ is cgt. if $x < 1$ and div. if $x > 1$.

When $x = 1$, $u_n = \dfrac{n+1}{n^2+1}$; take $v_n = \dfrac{1}{n}$.

Clearly $\dfrac{u_n}{v_n} \to 1$ and $\sum v_n$ is div. So by comparison test $\sum u_n$ is div.

(v) $\dfrac{u_n}{u_{n+1}} = \dfrac{n+1+\sqrt{1+(n+1)^2}}{n+\sqrt{1+n^2}} \cdot \dfrac{1}{x} \to 1.$

So by D'Alembert's test, $\sum u_n$ is cgt. if $x < 1$ and div. if $x > 1$.

When $x = 1$, $u_n = \dfrac{1}{n+\sqrt{1+n^2}}$; take $v_n = \dfrac{1}{n}$.

Clearly $\dfrac{u_n}{v_n} \to \dfrac{1}{2}$ and $\sum v_n$ is div. So by comparison test $\sum u_n$ is div.

(vi) $\dfrac{u_n}{u_{n+1}} = \sqrt{\dfrac{(n+1)\{(n+1)^3+1\}}{(n^3+1)(n+2)}} \cdot \dfrac{1}{x} \to \dfrac{1}{x}.$

So by D'Alembert's test $\sum u_n$ is cgt. if $x < 1$ and div. if $x > 1$.

When $x = 1$, $u_n = \sqrt{\dfrac{n+1}{n^3+1}}$; take $v_n = \dfrac{1}{n}$.

Clearly $\dfrac{u_n}{v_n} \to 1$ and $\sum u_n$ is div. So by comparison test $\sum u_n$ is div.

5. (i) $u_n = \dfrac{n^2-1}{n^2+1} x^n$;

So $\dfrac{u_n}{u_{n+1}} = \dfrac{n^2-1}{n^2+1} \cdot \dfrac{(n+1)^2+1}{(n+1)^2-1} \cdot \dfrac{1}{x} \to \dfrac{1}{x}.$

So by D'Alembert's test, $\sum u_n$ is cgt. if $x < 1$ and div. if $x > 1$.

When $x = 1$, $u_n = \dfrac{n^2 - 1}{n^2 + 1}$; take $v_n = 1$.

Clearly $\dfrac{u_n}{v_n} \to 1$ and Σv_n is div. So by comparison test Σu_n is div.

(or $\lim\limits_{n\to\infty} u_n = 1 \neq 0$. So Σu_n is div.)

(ii) $u_n = \dfrac{1}{1 + nx^n}$,

$$\text{So } \frac{u_n}{u_{n+1}} = \frac{1 + (n+1)x^{n+1}}{1 + nx^n} = \frac{\dfrac{1}{x^n(n+1)} + x}{\dfrac{1}{x^n(n+1)} + \dfrac{n}{n+1}}$$

$\therefore \lim\limits_{n\to\infty} \dfrac{u_n}{u_{n+1}} = x$ if $x > 1$.

So when $x > 1$, $\dfrac{u_n}{u_{n+1}} \to x > 1$.

So by D'Alembert's test Σu_n is cgt. when $x > 1$.

When $x < 1$, $\Sigma u_n = \Sigma \dfrac{1}{1 + nx^n} > \dfrac{1}{1+n} > \dfrac{1}{2n}$.

Clearly $\Sigma \dfrac{1}{2n}$ is div. So Σu_n is div. when $x < 1$.

When $x = 1$, $\Sigma u_n = \Sigma \dfrac{1}{1+n} = \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \ldots$ which is div.

6. Clearly it is a series of positive terms after some stage.

$u_n = \dfrac{a^{n-1}}{x + a^{n-1}}$; so $\dfrac{u_n}{u_{n+1}} = \dfrac{x + a^n}{x + a^{n-1}} \cdot \dfrac{1}{a}$.

If $a < 1$, $a^n \to 0$ and so $\dfrac{u_n}{u_{n+1}} \to \dfrac{x}{x} \cdot \dfrac{1}{a}$ i.e. $\dfrac{1}{a} > 1$.

$\therefore$ By D'Alembert's test, Σu_n is cgt. when $a < 1$.

When $a \geqslant 1$, $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \dfrac{a^{n-1}}{a^{n-1}\left(1+\dfrac{x}{a^{n-1}}\right)}$

$$= \lim_{n\to\infty} \frac{1}{1+\dfrac{x}{a^{n-1}}} = 1 \neq 0.$$

So $\sum u_n$ is div. for $a \geqslant 1$.

7. $u_n = \dfrac{x^{2n-2}}{(2n-1)(2n+1)}$;

So $\dfrac{u_n}{u_{n+1}} = \dfrac{(2n+1)(2n+3)}{(2n-1)(2n+1)} . \dfrac{1}{x^2} \to \dfrac{1}{x^2}$.

so by D'Alembert's test $\sum u_n$ is cgt. if $\dfrac{1}{x^2} > 1$ i.e. $x^2 < 1$

i.e. $x < 1$ or $x > -1$ while $\sum u_n$ is div. if $\dfrac{1}{x^2} < 1$

i.e. $x^2 > 1$ i.e. $x > 1$ or $x < -1$.

If $x^2 = 1$, $u_n = \dfrac{1}{4n^2 - 1}$; take $v_n = \dfrac{1}{n^2}$.

Clearly $\dfrac{u_n}{v_n} \to \dfrac{1}{4}$ and $\sum v_n$ is cgt. so, by comparison test $\sum u_n$ is cgt.

8. $u_n = \dfrac{n^n}{\underline{|n}}$; so $\dfrac{u_n}{u_{n+1}} = \dfrac{\underline{|n+1}}{\underline{|n}} . \dfrac{n^n}{(n+1)^{n+1}} = \dfrac{1}{\left(1+\dfrac{1}{n}\right)^n}$

So $\lim\limits_{n\to\infty} \dfrac{u_n}{u_{n+1}} = \dfrac{1}{e} < 1$.

By D'Alembert's test, $\sum u_n$ is div.

9. (i) $u_n = \left(1+\dfrac{1}{n}\right)^{-n^2}$;

So $u_n^{\frac{1}{n}} = \left(1+\dfrac{1}{n}\right)^{-n} = \dfrac{1}{\left(1+\dfrac{1}{n}\right)^n} \to \dfrac{1}{e} < 1$.

$\therefore$ By Cauchy's root test, $\sum u_n$ is cgt.

(ii) $u_n = \dfrac{1}{(\log n)^n}$; so $u_n^{\frac{1}{n}} = \dfrac{1}{\log n} \to 0 < 1$.

$\therefore$ By Cauchy's root test, $\sum u_n$ is cgt.

Hints to Problem Set 5

1. (i) $u_n = \dfrac{2.4.6.\dots(2n)}{3.5.7.\dots(2n+1)(2n+2)} x^{n-1}$;

$$\text{so} \quad \frac{u_n}{u_{n+1}} = \frac{(2n+3)(2n+4)}{(2n+2)^2} \cdot \frac{1}{x} \to \frac{1}{x}$$

So by D'Alembert's test, $\sum u_n$ is cgt. if $x < 1$ and div. if $x > 1$.

If $x = 1$, $n\left(\dfrac{u_n}{u_{n+1}} - 1\right) = \dfrac{3n^2 + 4n}{2(n+1)^2} \to \dfrac{3}{2} > 1$.

So by Raabe's test $\sum u_n$ is cgt.

(ii) $u_n = \dfrac{1.3.5.\dots(2n-3)}{2.4.6.\dots(2n-2)} \cdot \dfrac{x^n}{2n-1}$;

$$\text{so} \quad \frac{u_n}{u_{n+1}} = \frac{2n(2n+1)}{(2n-1)^2} \cdot \frac{1}{x} \to \frac{1}{x}.$$

So by D'Alembert's test, $\sum u_n$ is cgt. if $x < 1$ and div. if $x > 1$.

If $x = 1$, $n\left(\dfrac{u_n}{u_{n+1}} - 1\right) = \dfrac{6n^2 - n}{(2n-1)^2} \to \dfrac{6}{4} > 1$.

so by Raabe's test $\sum u_n$ is cgt.

(iii) Leaving the first term, $u_n = \dfrac{3.6.9.\dots 3n}{7.10.13.\dots(3n+4)} x^n$;

$$\text{so} \quad \frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3} \cdot \frac{1}{x} \to \frac{1}{x}.$$

So by D'Alembert's test $\sum u_n$ is cgt. if $x < 1$ and div. if $x > 1$.

If $x = 1$, $n\left(\dfrac{u_n}{u_{n+1}} - 1\right) = \dfrac{4n}{3n+3} \to \dfrac{4}{3} > 1$.

So by Raabe's test $\sum u_n$ is cgt.

2. (i) $u_n = \dfrac{2^2.4^2.6^2\dots.(2n-2)^2}{3.4.5\dots.2n}.x^{2n};$

$$\text{so}\quad \frac{u_n}{u_{n+1}} = \frac{(2n+1)(2n+2)}{(2n)^2}\,\frac{1}{x^2} \to \frac{1}{x^2}.$$

So by D'Alembert's test, Σu_n is cgt. if $x^2 < 1$
i.e. $x < 1$ or $x > -1$ while Σu_n is div. if $x^2 > 1$
i.e. $x > 1$ or $x < -1$.

$$\text{If } x^2 = 1,\ n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{3n+1}{2n} \to \frac{3}{2} > 1.$$

So by Raabe's test Σu_n is cgt.

(ii) Clearly it is a series of terms of the same sign.

Leaving the first term, $u_n = \dfrac{1.3.5\dots.(2n-1)}{2.4.6\dots.2n}.\dfrac{x^{2n+1}}{2n+1};$

$$\text{so}\quad \frac{u_n}{u_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2}\cdot\frac{1}{x^2} \to \frac{1}{x^2}.$$

So by D'Alembert's test, Σu_n is cgt. if $x^2 < 1$
i.e. $x < 1$ or $x > -1$ while Σu_n is div. if $x^2 > 1$
i.e. $x > 1$ or $x < -1$.

$$\text{If } x^2 = 1,\ n\left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{6n^2+5n}{(2n+1)^2} \to \frac{6}{4} > 1.$$

So by Raabe's test Σu_n is cgt.

3. (i) $u_n = \dfrac{n^{n-1}}{\lfloor n}\,x^{n-1};$

$$\text{so}\quad \frac{u_n}{u_{n+1}} = \frac{n^{n-1}}{\lfloor n}.\frac{\lfloor n+1}{(n+1)^n}\cdot\frac{1}{x}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n}.\left(1+\frac{1}{n}\right).\frac{1}{x} \to \frac{1}{ex}.$$

So by D'Alembert's test, Σu_n is cgt. if $\dfrac{1}{ex} > 1$

i.e. $x > \dfrac{1}{e}$ while Σu_n is div. if $x < \dfrac{1}{e}$.

If $x = \frac{1}{e}$, $n \log \frac{u_n}{u_{n+1}}$

$$= n\left[\log e + \log\left(1 + \frac{1}{n}\right) - n\log\left(1 + \frac{1}{n}\right)\right]$$

$$= n\left[1 + \left(\frac{1}{n} - \frac{1}{2n^2} + \dots\right) - n\left(\frac{1}{n} - \frac{1}{2n^2} + \dots\right)\right] \to \frac{3}{2} > 1.$$

$\therefore$ By logarithmic test, $\sum u_n$ is cgt.

(ii) $u_n = \frac{|\underline{n-1}}{n^{n-1}} x^{n-1}$;

so $$\frac{u_n}{u_{n+1}} = \frac{|\underline{n-1}}{n^{n-1}} \cdot \frac{(n+1)^n}{|\underline{n}} \cdot \frac{1}{x} = \left(1 + \frac{1}{n}\right)^n \cdot \frac{1}{x} \to \frac{e}{x}.$$

So by D'Alembert's test, $\sum u_n$ is cgt. if $\frac{e}{x} > 1$

i.e. $x < e$ and div. if $x > e$.

If $x = e$, $n \log \frac{u_n}{u_{n+1}} = -\frac{1}{2} + \frac{1}{3n} - \dots \to -\frac{1}{2} < 1.$

So by logarithmic test, $\sum u_n$ is div.

4. (i) $u_n = \frac{1^2.3^2.5^2\dots(2n-1)^2}{2^2.4^2.6^2\dots(2n)^2} x^{n-1}$;

so $$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} \cdot \frac{1}{x} \to \frac{1}{x}.$$

So by D'Alembert's test, $\sum u_n$ is cgt. if $\frac{1}{x} > 1$

i.e. $x < 1$ and div. if $x > 1$.

If $x = 1$, $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = 1$ and so Raabe's test fails.

Then $\lim_{n\to\infty} \left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n$

$$= \frac{-n-1}{4n^2+4n+1} \cdot \log n$$

$$= \frac{-n^2-n}{4n^2+4n+1} \cdot \frac{\log n}{n} \to \frac{-1}{4} . 0 = 0 < 1.$$

So by DeMorgan and Bertrand's test, $\sum u_n$ is div.

(ii) Omitting the first term,

$$u_n = \frac{\alpha(\alpha+1)^2(\alpha+2)^2 \ldots (\alpha+n-1)^2}{1.2.3 \ldots n . \beta(\beta+1)(\beta+2) \ldots (\beta+n-1)} . x^n.$$

So $\dfrac{u_n}{u_{n+1}} = \dfrac{(n+1)(\beta+n)}{(\alpha+n)^2} \cdot \dfrac{1}{x} \to \dfrac{1}{x}.$

Therefore, by D'Alembert's test, $\sum u_n$ is cgt. if $\dfrac{1}{x} > 1$

i.e. $x < 1$ and div. if $x > 1$.

If $x = 1$, $n\left(\dfrac{u_n}{u_{n+1}} - 1\right)$

$$= \frac{(\beta+1-2\alpha)n^2 + n(\beta-\alpha^2)}{(\alpha+n)^2} \to \beta + 1 - 2\alpha.$$

So by Raabe's test, $\sum u_n$ is cgt. if $\beta + 1 - 2\alpha > 1$
i.e. $\beta > 2\alpha$ and div. if $\beta < 2\alpha$.

If $\beta = 2\alpha$, $\left\{n\left(\dfrac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n$

$$= \frac{-n^2\alpha^2 - \alpha^2 n}{(\alpha+n)^2} \cdot \frac{\log n}{n} \to -\alpha^2 . 0 = 0 < 1.$$

So by DeMorgan and Bertrand's test $\sum u_n$ is div.

5. (i) $\dfrac{u_n}{u_{n+1}} = \dfrac{(n+a)(n+b)}{(n+1+a)(n+1+b)} \cdot \dfrac{n+3}{n} \to 1.$

So D'Alembert's test fails.

$\therefore\ n\left(\dfrac{u_n}{u_{n+1}} - 1\right)$

$$= \frac{n^2 + n(2a+2b-1) + 3ab}{n^3 + n^2(a+b+2) + n(1+a+b+ab)} \to 1.$$

So Raabe's test fails.

Now $\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n$

$$= \frac{a + b - 3 + \dfrac{2ab - 1 - a - b}{n}}{1 + \dfrac{a + b + 2}{n} + \dfrac{1 + a + b + ab}{n^2}} \cdot \frac{\log n}{n}$$

$$\to (a + b - 3) \times 0 = 0 < 1.$$

So by DeMorgan and Bertrand's test $\sum u_n$ is div.

(ii) $u_n = \dfrac{a(a + 1)(a + 2)\ldots.(a + n - 1)}{b(b + 1)(b + 2)\ldots.(b + n - 1)}$,

so $\dfrac{u_n}{u_{n+1}} = \dfrac{b + n}{a + n} \to 1.$

So D'Alembert's test fails.

Now $n\left(\dfrac{u_n}{u_{n+1}} - 1\right) = \dfrac{bn - an}{a + n} \to b - a.$

So by Raabe's test, $\sum u_n$ is cgt. if $b - a > 1$
i.e. $b > a + 1$ and div. if $b < a + 1$.

If $b = a + 1$, $\left\{n\left(\dfrac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n$

$$= \frac{-an}{a + n} \cdot \frac{\log n}{n} \to -a.0 = 0 < 1.$$

So by DeMorgan and Bertrand's test, $\sum u_n$ is div.

(iii) $u_n = \dfrac{\alpha(\alpha + 1)\ldots.(\alpha + n - 1)\beta(\beta + 1)\ldots.(\beta + n - 1)}{\gamma(\gamma + 1)\ldots.(\gamma + n - 1)\delta(\delta + 1)\ldots.(\delta + n - 1)}$,

So $\dfrac{u_n}{u_{n+1}} = \dfrac{(\gamma + n)(\delta + n)}{(\alpha + n)(\beta + n)} \to 1.$

So D'Alembert's test fails.

Now $n\left(\dfrac{u_n}{u_{n+1}} - 1\right)$

$$= \frac{n^2(\gamma + \delta - \alpha - \beta) + n(\gamma\delta - \alpha\beta)}{(\alpha + n)(\beta + n)} \to \gamma + \delta - \alpha - \beta.$$

So, by Raabe's test, $\sum u_n$ is cgt. if $\gamma + \delta - \alpha - \beta > 1$ and div. if $\gamma + \delta - \alpha - \beta < 1$.

If $\gamma + \delta - \alpha - \beta = 1$, we have $\left\{n\left(\dfrac{u_n}{u_{n+1}} - 1\right) - 1\right\}\log n$

$$= \frac{\gamma\delta - \alpha\beta - \alpha - \beta - \dfrac{\alpha\beta}{n}}{\left(1 + \dfrac{\alpha}{n}\right)\left(1 + \dfrac{\beta}{n}\right)} \cdot \frac{\log n}{n}$$

$$\to (\gamma\delta - \alpha\beta - \alpha - \beta).0 = 0 < 1.$$

So by DeMorgan and Bertrand's test, $\sum u_n$ is div.

6. (i) Here $p = 1$ in $\sum\limits_2^\infty \dfrac{1}{n(\log n)^p}$.

So the series is div. by auxiliary series of Cauchy condensation test.

(ii) Here $p = 3$ in $\sum\limits_2^\infty \dfrac{1}{n(\log n)^p}$.

So the series is cgt. by auxiliary series of Cauchy condensation test.

7. (i) $f(n) = \dfrac{\log n}{n}$ is positive decreasing as n increases faster than $\log n$. So $a^n f(a^n) = n \log a$.

But $\sum n$ is div. So $\sum a^n f(a^n)$ and hence $\sum f(n)$ is div. by Cauchy condensation test.

(ii) $f(n) = \dfrac{1}{\log n}$ is positive decreasing.

So $a^n f(a^n) = \dfrac{a^n}{n \log a} = u_n$ (say).

Now $\dfrac{u_n}{u_{n+1}} = \left(1 + \dfrac{1}{n}\right).\dfrac{1}{a} \to \dfrac{1}{a} < 1$ for $a > 1$.

So by D'Alembert's test, $\sum u_n$ is div. Hence, by Cauchy condensation test $\sum f(n)$ is div.

(iii) $u_{n+1} = \dfrac{\log n}{n^2 + 1} < \dfrac{\log n}{n^2} = f(n)$ which is positive decreasing.

So $2^n f(2^n) = \dfrac{\log 2}{2^n} . n = v_n$ (say).

Now $\dfrac{v_n}{v_{n+1}} = \dfrac{n}{n+1} . 2 \to 2 > 1.$

So $\sum 2^n f(2^n)$ is cgt. So by Cauchy condensation test $\sum f(n)$ is cgt. Hence by comparison test $\sum u_n$ is cgt.

8. $u_n = x^{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}}$ So $\dfrac{u_n}{u_{n+1}} = x^{-\frac{1}{n+1}} \to 1.$

So D'Alembert's test fails.

Now $n \log \dfrac{u_n}{u_{n+1}} = -\dfrac{n}{n+1} \log x \to \log \dfrac{1}{x}.$

So by logarithmic test $\sum u_n$ is cgt. if $\log \dfrac{1}{x} > 1$

i.e. $\dfrac{1}{x} > e$ i.e. $x < \dfrac{1}{e}$ and div. if $x > \dfrac{1}{e}.$

If $x = \dfrac{1}{e}$, $\left(n \log \dfrac{u_n}{u_{n+1}} - 1\right) \log n$

$$= \frac{-1}{1+\frac{1}{n}} . \frac{\log n}{n} \to -1 \times 0 = 0 < 1.$$

Hence by Higher logarithmic test $\sum u_n$ is div.

9. $u_n = \dfrac{2^2.4^2.6^2 \dots (2n)^2}{3^2.5^2.7^2 \dots (2n+1)^2}$;

so $\dfrac{u_n}{u_{n+1}} = \dfrac{(2n+3)^2}{(2n+2)^2} = \left(1+\dfrac{3}{2n}\right)^2 . \left(1+\dfrac{1}{n}\right)^{-2}$

$$= \left(1 + \frac{3}{n} + \frac{9}{4n^2}\right)\left(1 - \frac{2}{n} + \frac{3}{n^2} - \dots\right)$$

$\therefore \dfrac{u_n}{u_{n+1}} = 1 + \dfrac{1}{n} + \dfrac{1}{n^2}\left\{-\dfrac{23}{9} - \dfrac{13}{n} + \dots\right\}.$

By Gauss's test $\sum u_n$ is div. for here $\alpha = 1$.

Hints to Problem Set 6

1. First series is $\Sigma \frac{1}{n}$ i.e. $\Sigma \frac{1}{n^p}$ when $p = 1$. So it is div.

The second series is alternating for which $u_n \to 0$ and $u_n > u_{n+1}$. So by Leibnitz's rule it is cgt.

2. Here $u_n = \frac{1}{2n-1} \to 0$ and $\frac{u_n}{u_{n+1}} = \frac{2n+1}{2n-1} > 1$

i.e. $u_n > u_{n+1}$. So by Leibnitz's rule the series is cgt.

3. $\Sigma \mid u_n \mid = \Sigma \frac{1}{n^{\frac{1}{2}}}$ which is div. for $p = \frac{1}{2} < 1$ in $\Sigma \frac{1}{n^p}$.

But $u_n \to 0$ and $u_n > u_{n+1}$. So by Leibnitz's Rule the alternating series Σu_n is cgt. $\Sigma \mid u_n \mid$ is div., the given series is conditionally cgt.

4. It is an alternating series. Numerically, $u_n = \frac{1}{x+n} \to 0$.

and $u_n > u_{n+1}$ So by Leibnitz's rule, the series is cgt.

5. (i) The series is alternating and $u_n = \frac{a^{n-1}}{\underline{\lfloor n-1}} \to 0$ as

$n \to \infty$ for $0 < a < 1$, also $\frac{u_n}{u_{n+1}} = \frac{a^{n-1}}{\underline{\lfloor n-1}} \cdot \frac{\lfloor n}{a^n} = \frac{n}{a} > 1$

i.e. $u_n > u_{n+1}$. So by Leibnitz's rule, the series is cgt.

(ii) The series is alternating and $u_n = \frac{a^n}{1+a^n} \to 0$ for

$0 < a < 1$. Also $u_n - u_{n+1} = \frac{a^n(1-a)}{(1+a^n)(1+a^{n+1})} > 0$ for

$0 < a < 1$. So $u_n > u_{n+1}$. So by Leibnitz's rule, the series is cgt.

$$6.\ a_n = e^{-\frac{x}{n}} + \frac{x}{n} \cdot e^{-\frac{x}{n}} = \left(1 - \frac{x}{n} + \frac{1}{\lfloor 2} \frac{x^2}{n^2} - \dots\right)$$
$$+ \frac{x}{n}\left(1 - \frac{x}{n} + \frac{1}{\lfloor 2} \frac{x^2}{n^2} - \dots\right)$$

$$\therefore \quad v_n = a_n - 1 = -\frac{1}{n^2}\left(x^2 + \tfrac{1}{3}\cdot\frac{x^3}{n} + \ldots .\right)$$

take $V_n = \frac{1}{n^2}$. Clearly $\frac{|v_n|}{V_n} \to x^2 \neq 0$ when $x \neq 0$. But ΣV_n is cgt. So by comparison test $\Sigma\,|v_n|$ is cgt. i.e. $\Sigma\, v_n$ is absolutely convergent. When $x = 0$, the series vanishes.

7. (i) $u_n = \frac{n^n}{(2n-1)2n}$;

$$\therefore \quad \frac{|u_n|}{|u_{n+1}|} = \frac{(2n+1)(2n+2)}{(2n-1)2n}\cdot\left|\frac{1}{x}\right| \to \left|\frac{1}{x}\right|$$

So by D'Alembert's test, $\Sigma\,|u_n|$ is cgt. if $|x| < 1$. So $\Sigma\, u_n$ is absolutely cgt. and hence cgt. When $|x| < 1$ i.e. $x < 1$ or $x > -1$. If $|x| = 1$; $x = 1$ or -1. When $x = 1$, the series is $\frac{1}{1.2} + \frac{1}{3.4} + \ldots$. Then $u_n = \frac{1}{(2n-1)2n}$;

take $v_n = \frac{1}{n^2}$. Clearly $\frac{u_n}{v_n} \to \frac{1}{4}$ and $\Sigma\, v_n$ is cgt. So $\Sigma\, u_n$ is also cgt. (by comparison test).

If $x = -1$, the series becomes $\frac{1}{1.2} - \frac{1}{3.4} + \frac{1}{5.6} - \ldots .$ which is alternating and by Leibnitz's test it is cgt.

($\because$ $u_n \to 0$ and $u_n > u_{n+1}$).

If $|x| > 1$, $x > 1$ or $x < -1$.

In case $x > 1$, it is a series of positive terms and

$$\frac{u_n}{u_{n+1}} = \frac{(2n+1)(2n+2)}{(2n-1)2n}\cdot\frac{1}{x} \to \frac{1}{x} < 1.$$

So by D'Alembert's test $\Sigma\, u_n$ is div.

In case $x < -1$, let $x = -y$ and so $y > 1$ and the series becomes $\frac{-y}{1.2} + \frac{y^2}{3.4} - \frac{y^3}{5.6} + \ldots .$

$$\therefore \quad |u_n| = \frac{y^n}{(2n-1)2n} > \frac{y^n}{4n^2};$$ as $n \log y > 8 \log n$ after some stage (as $y > 1$) we get $y^n > 4n^2$ after some stage. So $|u_n|$ does not tend to zero. Hence the series is not convergent.

(ii) $u_n = \dfrac{x^{n-1}}{2n-1}$; $\therefore$ $\dfrac{|u_n|}{|u_{n+1}|} = \dfrac{2n+1}{2n-1} \cdot \left|\dfrac{1}{x}\right| \to \left|\dfrac{1}{x}\right|$

So by D'Alembert's test $\Sigma\, |u_n|$ is cgt. if $|x| < 1$.

So $\Sigma\, u_n$ is absolutely cgt. and hence cgt. when $|x| < |$ i.e. $x < 1$ or $x > -1$.

If $|x| = 1, x = 1$ or -1.

When $x = 1$, the series is $1 + \frac{1}{3} + \frac{1}{5} + \ldots\ldots$

So $u_n = \dfrac{1}{2n-1}$; take $v_n = \dfrac{1}{n}$.

Clearly $\Sigma\, v_n$ is div. and so $\Sigma\, u_n$ is divergent.

When $x = -1$, the series is $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$

which is cgt. by Leibnitz's rule.

If $|x| > 1, x > 1$ or $x < -1$.

When $x > 1, \dfrac{u_n}{u_{n+1}} = \dfrac{2n+1}{2n-1} \cdot \dfrac{1}{x} \to \dfrac{1}{x} < 1$.

So by D'Alembert's test, $\Sigma\, u_n$ is div.

If $x < -1$ let $y = -x$ and $y > 1$.

Then the series is $1 - \frac{y}{3} + \frac{y^2}{5} - \frac{y^3}{7} + \ldots\ldots$

Then $|u_n| = \dfrac{y^{n-1}}{2n-1} > \dfrac{y^{n-1}}{2n}$; as $(n-1)\log y > \log 2n$ after some stage (as $y > 1$) we get $y^{n-1} > 2n$ after some stage. So $|u_n|$ does not tend to zero.

Hence the series is not convergent.

(iii) $|u_n| = \left|\dfrac{\cos nx}{n^k}\right| \leqslant \left|\dfrac{1}{n^k}\right|$.

But $\Sigma \left|\dfrac{1}{n^k}\right|$ is cgt if $k > 1$.

So by comparison test, $\Sigma\, |u_n|$ is cgt. and so Σu_n is absolutely cgt.

8. (i) $| u_n | = \left| \frac{x^n}{n^{\frac{1}{2}}} \right|$;

so $\left| \frac{u_n}{u_{n+1}} \right| = \left| \frac{(n+1)^{\frac{1}{2}}}{n^{\frac{1}{2}}} \right| \left| \frac{1}{x} \right| \rightarrow \frac{1}{| x |}$.

So by D'Alembert's test the series is absolutely cgt. and hence cgt. if $| x | < 1$.

If $| x | = 1, x = 1$ or -1.

When $x = 1$, the series is $-1 + \frac{1}{2^{\frac{1}{2}}} - \frac{1}{3^{\frac{1}{2}}} + \ldots.$for which $u_n \rightarrow 0$ and $u_n > u_{n+1}$ (numerically).

So by Leibnitz's rule the series is cgt.

But $\Sigma \, | u_n |$ is div. for $p = \frac{1}{2} < 1$ in $\Sigma \frac{1}{n^p}$.

Hence the series is conditionally cgt. when $x = 1$.

When $x = -1$, the series is $1 + \frac{1}{2^{\frac{1}{2}}} + \frac{1}{3^{\frac{1}{2}}} + \ldots$which is div.

If $| x | > 1$, $x > 1$ or $x < -1$.

When $x < -1$. let $x = -y$.

So $y > 1$ and the series is $\frac{y}{\sqrt{1}} + \frac{y^2}{\sqrt{2}} + \frac{y^3}{\sqrt{3}} + \ldots.$ which is a series of positive terms and

$$\frac{u_n}{u_{n+1}} = \left(\frac{n+1}{n} \right)^{\frac{1}{2}} \cdot \frac{1}{y} \rightarrow \frac{1}{y} < 1.$$

So by D'Alembert's test the series is div.

When $x > 1$, $n \log x > \frac{1}{2} \log n$ after some stage.

So $x^n > \sqrt{n}$ after some stage. So $| u_n | > 1$ after some stage.

Therefore u_n does not tend to zero.

Hence the series is not cgt.

(ii) It is alternating, $| u_n | = \frac{\log n}{\sqrt{n}}$;

So numerically, $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{\log n}{\sqrt{n}} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{3}}} = 0$

and u_n is decreasing as $\sqrt{n}$ increases at a faster rate than $2 \log \sqrt{n}$.

So by Leibnitz's rule, the series is cgt.

9. See 3.1 when $x = 1$; here take $x - 1$ in place of n.

Hints to Problem Set 7

1. $\lim\limits_{h\to 0} f(0 + h) = \lim\limits_{h\to 0} \dfrac{h}{h} = 1$; $\lim\limits_{h\to 0} f(0 - h) = \lim\limits_{h\to 0} \dfrac{h}{-h}$ $= -1$. So the limit does not exist.

2. $\lim\limits_{h\to 0} f(0 + h) = \lim\limits_{h\to 0} (0 + h)^2 = 0$;

$\lim\limits_{h\to 0} f(0 - h) = \lim\limits_{h\to 0} (0 - h) = 0$; so the limit exists.

3. (i) $\lim\limits_{h\to 0} f(0 + h) = \lim\limits_{h\to 0} h \sin 1/h = 0$;

$\lim\limits_{h\to 0} f(0 - h) = -h \sin \dfrac{1}{-h} = 0; f(0) = 1$; as $0 = 0 \neq 1$,

$f(x)$ is discontinuous at $x = 0$.

(ii) Similar to worked out 2.

4. (i) $\lim\limits_{h\to 0} f(0 - h) = \lim\limits_{h\to 0} -(0 - h)^2 = 0$;

$\lim\limits_{h\to 0} f(0 + h) = \lim\limits_{h\to 0} [5(0 + h) - 4] = -4$; $f(0) = -0^2$;

So discontinuous at $x = 0$.

$\lim\limits_{h\to 0} f(1 - h) = \lim\limits_{h\to 0} [5(1 - h) - 4] = 1$;

$\lim\limits_{h\to 0} f(1 + h) = \lim\limits_{h\to 0} [4(1 + h)^2 - 3(1 + h)] = 1$;

$f(1) = 5.1 - 4 = 1$; so continuous at $x = 1$.

$\lim\limits_{h\to 0} f(2 - h) = \lim\limits_{h\to 0} [4(2 - h)^2 - 3(2 - h)] = 10$;

$\lim\limits_{h\to 0} f(2 + h) = \lim\limits_{h\to 6} [3(2 + h) + 4] = 10$;

$f(2) = 3.2 + 4 = 10$; so continuous at $x = 2$.

(ii) Similar to above.

5. $\lim\limits_{h\to 0} f(a - h) = \lim\limits_{h\to 0} \left[\dfrac{(a - h)^2}{a^3} - a \right] = 0$;

$\lim\limits_{h\to 0} f(a + h) = \lim\limits_{h\to 0} \left[a - \dfrac{a^3}{(a + h)^2} \right] = 0$; $f(a) = 0$;

so continuous at $x = a$.

6. $\lim_{h\to 0} f(0-h) = \lim_{h\to 0} 2^{-\frac{1}{h}} = \lim_{h\to 0} \frac{1}{2^{1/h}} = 0.$

$\lim_{h\to 0} f(0+h) = \lim_{h\to 0} 2^{\frac{1}{h}} = \infty$; So discontinuous at $x = 0$.

7. (i) $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} [0+h] = 0$; $f(0) = 1$;

so discontinuous at $x = 0$.

$\lim_{h\to 0} f(1-h) = \lim_{h\to 0} [1-h] = 1$; $f(1) = 0$;

so discontinuous at $x = 1$.

(ii) Similar to (i).

8. $\lim_{h\to 0} f(a-h) = \lim_{h\to 0} \frac{(a-h)^2 - a^2}{a-h-a}$

$$= \lim_{h\to 0} (-h+2a) = 2a.$$

$\lim_{h\to 0} f(a+h) = \lim_{h\to 0} 2(a+h) = 2a$; $f(a) = 2a$.

so it is continuous at $x = a$.

9. $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{e^{\frac{1}{h}}}{1+e^{\frac{1}{h}}} = 1$;

$\lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{e^{-\frac{1}{h}}}{1+e^{-\frac{1}{h}}} = 0$; $f(0) = 0$.

So it is discontinuous at $x = 0$.

10. (i) $\lim_{h\to 0} f(0 \pm h) = \lim_{h\to 0} \frac{3(0\pm h) + 4\tan(\pm h)}{\pm h}$

$$= \lim_{h\to 0} \left[3 + 4\frac{\tan(\pm h)}{\pm h}\right] = 3 + 4.1 = 7;$$

$f(0)$ is not defined.

So it is not continuous at $x = 0$.

If $f(0) = 7$, then it is continuous at $x = 0$.

(ii) Similar to (i).

11. (i) $\lim\limits_{h\to 0} f(0-h) = \lim\limits_{h\to 0} [3 + 2(0-h)] = 3;$

$\lim\limits_{h\to 0} f(0+h) = \lim\limits_{h\to 0} [3 - 2(0+h)] = 3;\ f(0) = 3.$

So $f(x)$ is continuous at $x = 0$.

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{(3-2h) - 3}{h} = -2.$$

$$\lim_{h\to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h\to 0} \frac{3 - 2h - 3}{-h} = 2.$$

So it is not differentiable at $x = 0$.

(ii) $\lim\limits_{h\to 0} f(0 \pm h) = \lim\limits_{h\to 0} (\pm h) \sin \dfrac{1}{\pm h} = 0;\ f(0) = 0.$

So it is continuous at $x = 0$.

$$\lim_{h\to 0} \frac{f(0 \pm h) - f(0)}{\pm h} = \lim_{h\to 0} \frac{(\pm h) \sin \dfrac{1}{\pm h} - 0}{\pm h} = \lim_{h\to 0} \sin \frac{1}{\pm h}$$

which is not definite. So it is not differentiable at $x = 0$.

12. $$\lim_{h\to 0} \frac{f(0 \pm h) - f(0)}{\pm h} = \lim_{h\to 0} \frac{(0 \pm h)^2 \cos \dfrac{1}{\pm h}}{\pm h}$$

$$= \lim_{h\to 0} \pm h \cos \frac{1}{\pm h} = 0.$$

$\therefore$ $f'(0)$ exists and it is 0.

13. $\lim\limits_{h\to 0} f(1-h) = \lim\limits_{h\to 0} [1 - h - 1] = 0;$

$\lim\limits_{h\to 0} f(1+h) = \lim\limits_{h\to 0} [1 - (1+h)] = 0;$

$f(0) = 0$ and so it is continuous at $x = 1$.

Now $\lim\limits_{h\to 0} \dfrac{f(1-h) - f(1)}{-h} = \lim\limits_{h\to 0} \dfrac{-h-0}{-h} = 1;$

$$\lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{-h-0}{h} = -1.$$

So it is not differentiable at $x = 1$.

14. (i) $\lim\limits_{h\to 0} \dfrac{f(0 \pm h) - f(0)}{\pm h} = \lim\limits_{h\to 0} \dfrac{e^{-\frac{1}{(\pm h)^2}} . \sin \dfrac{1}{\pm h}}{\pm h}$

$$= \lim_{h\to 0} \frac{\sin \dfrac{1}{\pm h}}{\pm h .\, e^{\frac{1}{h^2}}} = \lim_{h\to 0} \frac{\sin \dfrac{1}{\pm h}}{\pm h \left[1 + \dfrac{1}{h^2} + \dfrac{1}{\lfloor 2 h^4} + \dots\right]}$$

$$= \lim_{h\to 0} \frac{\sin \dfrac{1}{\pm h}}{\pm \left[h + \dfrac{1}{h} + \dots\right]} = \frac{\text{number between } 1 \;\&\; -1}{\infty} = 0.$$

$\therefore$ It is differentiable at $x = 0$.

(ii) $\lim\limits_{h\to 0} \dfrac{f\left(\dfrac{\pi}{2} + h\right) - f\left(\dfrac{\pi}{2}\right)}{h}$

$$= \lim_{h\to 0} \frac{2 + \left(\dfrac{\pi}{2} + h - \dfrac{\pi}{2}\right)^2 - \left[2 + \left(\dfrac{\pi}{2} - \dfrac{\pi}{2}\right)^2\right]}{h} = 0,$$

$$\lim_{h\to 0} \frac{f\left(\dfrac{\pi}{2} - h\right) - f\left(\dfrac{\pi}{2}\right)}{-h} = \lim_{h\to 0} \frac{1 + \sin\left(\dfrac{\pi}{2} - h\right) - 2}{-h}$$

$$= \lim_{h\to 0} \frac{1 - \cos h}{h} = 0.$$

$\therefore$ It is differentiable at $x = \dfrac{\pi}{2}$.

15. (i) $\lim\limits_{h\to 0} f(x \pm h) = \lim\limits_{h\to 0} [(x \pm h)^3 - 3(x \pm h)] = x^3 - 3x;$
$f(x) = x^3 - 3x.$

$\therefore$ The function is continuous at all real x.

(ii) $\lim\limits_{h\to 0} f(a \pm h) = \lim\limits_{h\to 0} \sqrt{5(a \pm h) - (a \pm h)^2 - 6}$

$$= \sqrt{5a - a^2 - 6}\,;$$

$f(a) = \sqrt{5a - a^2 - 6}$. So it is continuous for all $x = a$.

16. Let $\varphi(x) = f'(x)$.

Here $\varphi(x) = 0,\ 0 \leqslant x \leqslant a$

$$= -\tfrac{1}{3}x + \tfrac{1}{3}\cdot\frac{a^3}{x^2},\quad a < x \leqslant 2a$$

$$= -\tfrac{7}{3}\,\frac{a^3}{x^2},\ x > 2a.$$

Now, $\lim\limits_{h\to 0} \varphi(0 + h) = \lim\limits_{h\to 0} 0 = 0;\ \varphi(0) = 0.$

So $\varphi(x)$ is continuous at $x = 0$. Let $0 < c < a$.

$\lim\limits_{h\to 0} f(c \pm h) = \lim\limits_{h\to 0} 0 = 0;\ \ f(c) = 0.$

So $\varphi(x)$ is continuous in $0 < c < a$.

$$\lim_{h\to 0} f(a + h) = \lim_{h\to 0}\left[-\frac{1}{3}\left(a + h\right) + \frac{1}{3}\cdot\frac{a^3}{(a + h)^2}\right] = 0;$$

$\lim\limits_{h\to 0} f(a - h) = \lim\limits_{h\to 0} 0 = 0;\ \ f(a) = 0.$

So $\varphi(x)$ is continuous at $x = a$.

Let $a < c < 2a$,

$$\lim_{h\to 0} f(c \pm h) = \lim_{h\to 0}\left[-\frac{1}{3}\left(c \pm h\right) + \frac{1}{3}\cdot\frac{a^3}{(c \pm h)^2}\right]$$

$$= -\frac{1}{3}\,c + \frac{1}{3}\,\frac{a^3}{c^2} \text{ and } f(c) = -\frac{1}{3}c + \frac{1}{3}\,\frac{a^3}{c^2}$$

So $\varphi(x)$ is continuous in $a < x < 2a$.

$$\lim_{h\to 0} \varphi(2a + h) = \lim_{h\to 0}\left[-\frac{7}{3}\,\frac{a^3}{(2a + h)^2}\right] = -\frac{7}{3}\cdot\frac{a}{4};$$

$$\lim_{h\to 0} \varphi(2a - h) = \lim_{h\to 0}\left[-\frac{1}{3}\left(2a - h\right) + \frac{1}{3}\cdot\frac{a^3}{(2a - h)^2}\right]$$

$$= \frac{-7a}{12};\ \text{also } \varphi(2a) = -\frac{1}{3}\cdot 2a + \frac{1}{3}\cdot\frac{a^2}{(2a)^2} = -\frac{7a}{12}.$$

So $\varphi(x)$ is continuous at $x = 2a$.

Let $c > 2a$. Then $\lim\limits_{h\to 0} \varphi(c \pm h) = \lim\limits_{h\to 0} \left[-\frac{7}{3}\, \frac{a^3}{(c \pm h)^2} \right]$

$$= -\frac{7}{3} \cdot \frac{a^3}{c^2} \text{ and } \varphi(c) = -\frac{7}{3}\frac{a^3}{c^2}.$$

So $\varphi(x)$ is continuous for $x > 2a$.

17. Clearly the function is differentiable at $x = a \neq 1, 0$

Now $\lim\limits_{h\to 0} \dfrac{f(1 \pm h) - f(1)}{\pm h}$

$$= \lim_{h\to 0} \frac{\left\{\left(\pm h\right)^2 \sin \frac{1}{\mp h} - |1 \pm h|\right\} - \left(-1\right)}{\pm h}$$

$$= \lim_{h\to 0} \frac{\mp h^2 . \sin \frac{1}{h} - (1 \pm h) + 1}{\pm h}, \text{ for } h \text{ is small positive quantity}$$

$$= \lim_{h\to 0} \left[-h . \sin \frac{1}{h} - 1 \right] = -1.$$

$\therefore$ $f(x)$ is differentiable every where except perhaps 0.

Next $\lim\limits_{h\to 0} \dfrac{f(0 \pm h) - f(0)}{\pm h}$

$$= \lim_{h\to 0} \frac{\left\{\left(\pm h - 1\right)^2 . \sin \frac{1}{1 \mp h} - |\pm h|\right\} - \sin 1}{\pm h}$$

$$= \lim_{h\to 0} \left[\left(\pm h - 2 \mp \frac{1}{h}\right) \sin \frac{1}{1 \mp h} \mp 1 \mp \sin 1 \right] = \mp \infty.$$

$\therefore$ The set of pts. where $f(x)$ is not differentiable is $\{0\}$.

18. (i) Let $-1 \leqslant c < 0$.

$\lim\limits_{h\to 0} f(c \pm h) = \lim\limits_{h\to 0} [1 - (c \pm h)^2] = 1 - c^2$, $f(c) = 1 - c^2$;

So $f(x)$ is continuous for $-1 \leqslant x < 0$.

$\lim\limits_{h\to 0} f(0 + h) = \lim\limits_{h\to 0} \cos(0 + h) = 1$;

$\lim\limits_{h\to 0} f(0 - h) = \lim\limits_{h\to 0} [1 - (0 - h)^2] = 1$; $f(0) = 1 - 0^2 = 1$;

So $f(x)$ is continuous at $x = 0$.

Let $0 < c' \leqslant \frac{\pi}{2}$; $\lim\limits_{h\to 0} f(c' \pm h) = \lim\limits_{h\to 0} \cos(c' \pm h) = \cos c'$;

$f(c^r) = \cos c'$.

So $f(x)$ is continuous for $0 < x \leqslant \frac{\pi}{2}$.

Then $f(x)$ is continuous for $-1 \leqslant x < 0$ as well as $x = 0$ and $0 < x \leqslant \frac{\pi}{2}$ i.e. for $-1 \leqslant x \leqslant \frac{\pi}{2}$.

(ii) Let $-1 < c < 0$.

$$\lim_{h\to 0} \frac{f(c \pm h) - f(c)}{\pm h} = \lim_{h\to 0} \frac{[1-(c \pm h)^2] - (1-c^2)}{\pm h} = -2c.$$

So $f(x)$ is differentiable in $-1 < x < 0$.

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\cos h - 1}{h} = 0$$

and $\lim\limits_{h\to 0} \dfrac{f(0-h) - f(0)}{-h} = \lim\limits_{h\to 0} \dfrac{[1-(-h)^2] - 1}{-h} = 0.$

So $f(x)$ is differentiable at $x = 0$.

Let $0 < c < \frac{\pi}{2}$. Then

$$\lim_{h\to 0} \frac{f(c \pm h) - f(c)}{\pm h} = \lim_{h\to 0} \frac{\cos(c \pm h) - \cos c}{\pm h} = -\sin c.$$

So $f(x)$ is differentiable in $0 < x < \frac{\pi}{2}$.

Hence $f(x)$ is differentiable in $-1 < x < 0$, at $x = 0$ and in $0 < x < \frac{\pi}{2}$.

$\therefore$ $f(x)$ is differentiable in $-1 < x < \frac{\pi}{2}$.

19. $\lim\limits_{h\to 0} \varphi(0 \pm h) = \lim\limits_{h\to 0} |\pm h| + |\pm h - 1| = 1 = \varphi(0)$.

$\lim\limits_{h\to 0} \varphi(1 \pm h) = \lim\limits_{h\to 0} |1 \pm h| + |1 \pm h - 1| = 1 = \varphi(1)$.

So $\varphi(x)$ is continuous at $x = 0, 1$

$$\lim_{h\to 0}\frac{\varphi(0 \pm h)-\varphi(0)}{\pm h}$$

$$= \lim_{h\to 0}\frac{|\pm h| + |\pm h - 1| - 1}{\pm h} = 0, 2.$$

$$\lim_{h\to 0}\frac{\varphi(1 \pm h)-\varphi(1)}{\pm h} = \lim_{h\to 0}\frac{|1 \pm h| + |\pm h| - 1}{\pm h} = 2, 0$$

∴ $\varphi(x)$ is not derivable for $x = 0, 1$,

20. As continuous at $x = 0$, $\lim\limits_{h\to 0} f(0 \pm h) = f(0)$.

Now $\lim\limits_{h\to 0} f(a \pm h) = \lim\limits_{h\to 0} [f(a) + f(\pm h)]$,

(by given condition

$= f(a) + \lim\limits_{h\to 0} [f(0 \pm h)] = f(a) + f(0) = f(a + 0) = f(a)$.

Problem Set 8

1. Let $f(x) = x^5 - 3x - 1$; clearly $f(x)$ is a continuous function R, being a polynomial function. Also $f(1) = -3$, $f(2) = 25$. So. $f(2)$ are of opposite signs. Hence at least one root of $f(x) = 0$ between 1 and 2.

2. $f(x) = a_1(x-\lambda_2)(x-\lambda_3) + a_2(x-\lambda_1)(x-\lambda_3) + a_3(x-\lambda_1)(x-\lambda_2) =$

∴ $f(\lambda_1) = a_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3) = (+)(+)(+)$

$f(\lambda_2) = a_2(\lambda_2-\lambda_1)(\lambda_2-\lambda_3) = (+)(+)(-)$

$f(\lambda_3) = a_3(\lambda_3-\lambda_1)(\lambda_3-\lambda_2) = (+)(+)(+)$

As $f(\lambda_1) > 0$, $f(x_2) < 0$ one root of $f(x) = 0$ lies between λ_1, λ_2.

As $f(\lambda_2) < 0$, $f(\lambda_3) > 0$ one root of $f(x) = 0$ lies between λ_2, λ_3.

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